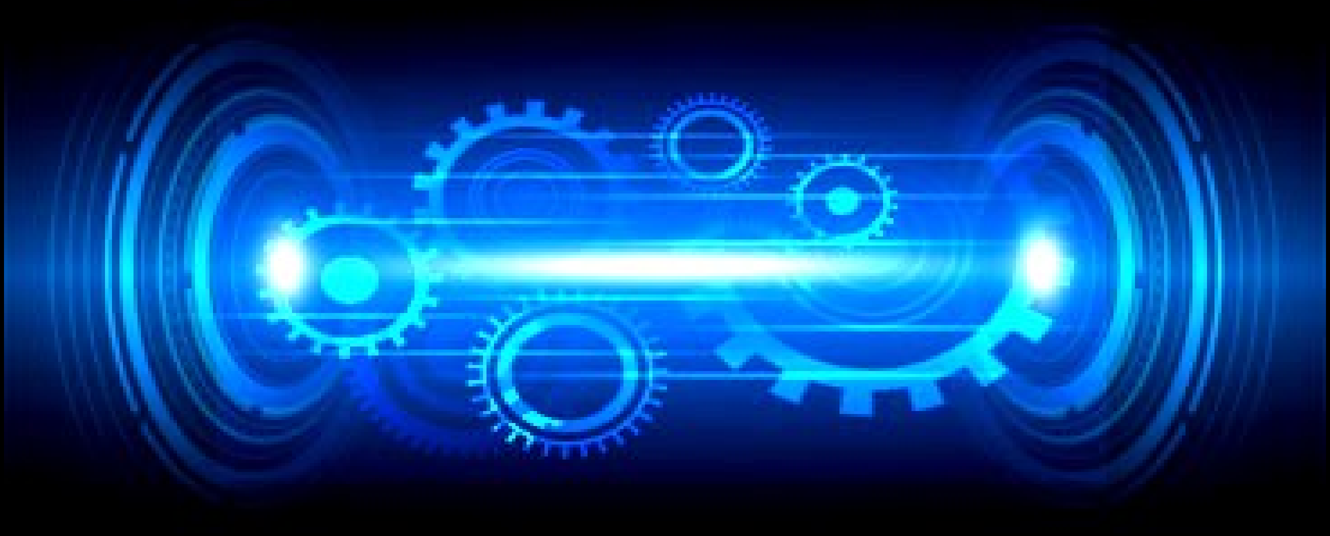
PHYSICS

11th

for JEE, NEET & BOARD

MOTION OF SYSTEM OF PARTICLES & RIGID BODY



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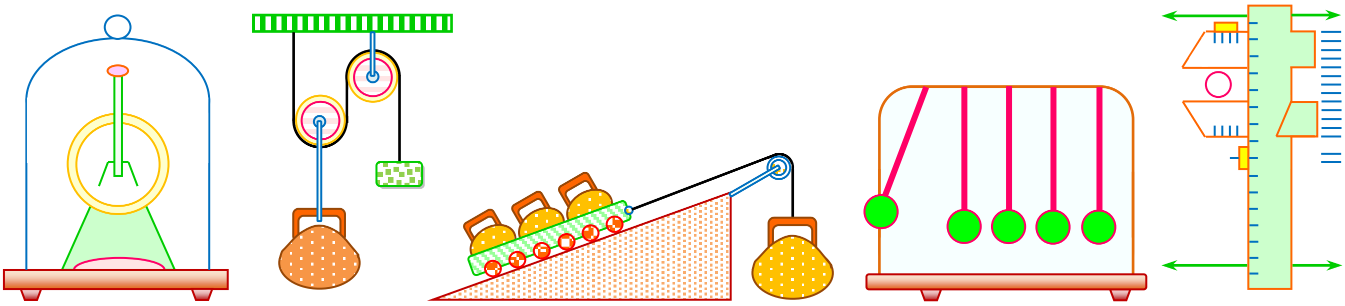

PHYSICS BOOKLET FOR JEE NEET & BOARDS

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MOTION OF SYSTEM OF PARTICLES & RIGID BODY

1 SYSTEMS OF PARTICLES: CENTRE OF MASS

Until now we have dealt mainly with single particle. Bodies like block, man, car etc. are also treated as particles while describing its motion. The particle model was adequate since we were concerned only with translational motion. When the motion of a body involves rotation and vibration, we must treat it as a system of particles. In spite of complex motion of which a system is capable, there is a single point, the centre of mass (CM), whose translational motion is characteristic of the system as a whole. Here we shall discuss about location of centre of mass of a system of particles and its motion.

1.1 CENTRE OF MASS OF CONTINUOUS BODIES

For calculating centre of mass of a continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice is usually determined by the symmetry of body.

Consider element *dm* of the body having position vector $\vec{r}$, the quantity $m_i\vec{r}_i$ in equation of CM is replaced by $\vec{r}\,dm$ and the discrete sum over particles $\frac{\Sigma m_i r_i}{M}$, becomes integral over the body:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm \qquad \ldots(1)$$

In component form this equation can be written as

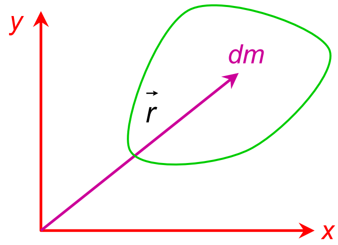


Fig. (1)

$$X_{CM} = \frac{1}{M}\int x\,dm \;;\; Y_{CM} = \frac{1}{M}\int y\,dm \text{ and } Z_{CM} = \frac{1}{M}\int Z\,dm \qquad \ldots(2)$$

To evaluate the integral we must express the variable *m* in terms of spatial coordinates *x*, *y*, *z* or $\vec{r}$.

Illustration 1

Question:

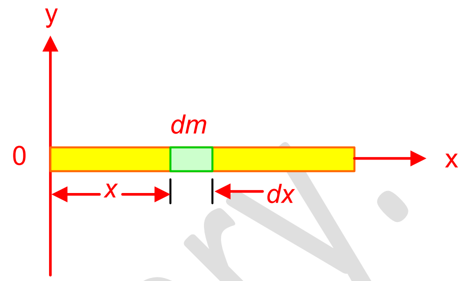
(a) **Show that the centre of mass of a rod of mass *M* and length *L*= 6m lies midway between its ends, assuming the rod has a uniform mass per unit length.**

(b) **Suppose a rod is non-uniform such that its mass per unit length varies linearly with *x* according to the expression $\lambda = \alpha x$, where α is a constant. Find the *x* coordinate of the centre of mass as a fraction of *L*.**

Solution:

(a) By symmetry, we see that $y_{CM} = z_{CM} = 0$ if the rod is placed along the x axis. Furthermore, if we call the mass per unit length λ (the linear mass density), then $\lambda = M/L$ for a uniform rod. If we divide the rod into elements of length dx, then the mass of each element is $dm = \lambda\, dx$. Since an arbitrary element of each element is at a distance x from the origin, equation gives

$$x_{CM} = \frac{1}{M}\int_0^L x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda L^2}{2M}$$



Because λ = M/L, this reduces to

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \frac{L}{2} = \mathbf{3m}$$

One can also argue that by symmetry, $x_{CM} = L/2$.

(b) In this case, we replace dm by $\lambda\, dx$, where λ is not constant. Therefore, x_{CM} is

$$x_{CM} = \frac{1}{M}\int_0^L x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\alpha}{M}\int_0^L x^2\, dx = \frac{\lambda L^3}{3M}$$

We can eliminate α by noting that the total mass of the rod is elated to α through the relationship

$$M = \int dm = \int_0^L \lambda\, dx = \int_0^L \alpha x\, dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

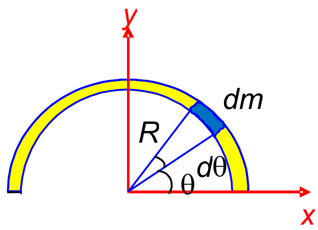
$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L = \mathbf{4m}$$

Illustration 2

Question: **Locate the centre of mass of a uniform semicircular rod of radius $R = \pi$ m and linear density λ kg/m.**

Solution:

From the symmetry of the body we see at once that the CM must lie along the y axis, so $x_{CM} = 0$. In this case it is convenient to express the mass element in terms of the angle θ, measured in radians. The element, which subtends an angle $d\theta$ at the origin, has a length $R\, d\theta$ and a mass $dm = \lambda\, Rd\theta$. Its y coordinate is $y = R \sin\theta$.



Therefore, $y_{CM} = \int \frac{y\, dm}{M}$ takes the

$$y_{CM} = \frac{1}{M}\int_0^\pi \lambda R^2 \sin\theta\, d\theta = \frac{\lambda R^2}{M}[-\cos\theta]_0^\pi = \frac{2\lambda R^2}{M}$$

The total mass of the ring is $M = \pi R\lambda$; therefore, $y_{CM} = \frac{2R}{\pi} = \mathbf{2\ m}$.

1.2 EQUATION OF MOTION FOR A SYSTEM OF PARTICLES

Acceleration of centre of mass $\vec{a}_{CM}$ is given by $\vec{a}_{CM} = \frac{\Sigma m_i \vec{a}_i}{\Sigma m_i} = \frac{1}{M}\Sigma m_i .\vec{a}_i$

Rearranging the expression and using Newton's second law, we get

$$M\vec{a}_{CM} = \Sigma m_i \vec{a}_i = \Sigma \vec{F}_i$$

where $\vec{F}_i$ is the force on i th particle.

The force on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However by Newton's third law, the force exerted by particle 1 on particle 2, for example, is equal to and opposite the force exerted by particle 2 on particle 1. Thus, when we sum over all internal forces in above equation they cancel in pairs and the net force is only due to external forces. Thus we can write equation of motion of centre of mass in the form.

$$\Sigma \vec{F}_{ext} = M\vec{a}_{CM} \quad \ldots(3)$$

Thus the acceleration of the centre of mass of a system is the same as that of a particle whose mass is total mass of the system, acted upon by the resultant external forces acting on the system.

If $\Sigma \vec{F}_{ext} = 0$, then centre of mass of system will move with uniform speed and if initially it were at rest it will remains at rest.

Illustration 3

Question: **A man weighing 70 kg is standing at the centre of a flat boat of mass 350 kg. The man who is at a distance of 10 m from the shore walks 2 m towards it and stops. How far will he be from the shore? Assume the boat to be of uniform thickness and neglect friction between boat and water.**

Solution: Consider that the boat and the man on it constitute a system. Initially, before the man started walking, the centre of mass of the system is at 10 m away from the shore and is at the centre of the boat itself. The centre of mass is also initially at rest.

As no external force acts on this system, the centre of mass will remain stationary at this position. Let us take this point as the origin and the direction towards the shore as x-axis.

If x_1 and x_2 be the position coordinates of man and centre of boat respectively, at any instant, position coordinate of the centre of mass

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

i.e., $$0 = \frac{70x_1 + 350x_2}{70 + 350}$$

$$x_1 + 5x_2 = 0 \quad \ldots\text{(i)}$$

Also, $$x_1 - x_2 = 2 \quad \ldots\text{(ii)}$$

Solving equations (i) and (ii) $x_1 = \frac{5}{3}m$

Since the centre of mass of the system remains stationary the man will be at a distance $10 - \frac{5}{3} =$ **830 cm** from the shore.

2 MOTION OF RIGID BODIES

A rigid body is a body whose deformation is negligible when subjected to external forces. In a rigid body the distance between any two points remains constant. A rigid body can undergo various types of motion. It may translate, rotate or may translate and rotate at the same time.

When a rigid body translates each particle of rigid body undergoes same displacement,, has same velocity and same acceleration. To apply equation of translation, all of its mass can be assumed to be concentrated at its center of mass and we can use $\vec{F}_{ext} = M\vec{a}_{CM}$. So our study takes the form as we study in case of particle dynamics.

If a body rotates or if it translates and rotates simultaneously, the case is different and we shall study it separately in details.

2.1 MOMENT OF INERTIA

One of the most fundamental characteristics possessed by an object is its intrinsic reluctance to accept a change in its state of motion i.e., its inertia.

A body needs a force to start its translation motion and its translational inertia is better known as mass. Also force is directly proportional to mass of body and linear acceleration of body.

On the other hand the state of motion of a body can undergo change in rotation if a torque is applied. The resulting angular acceleration depends partly on the magnitude of the applied torque, however the same torque applied to different bodies produce different angular acceleration, indicating that each body has an individual amount of rotational inertia which controls the degree of change in motion. The measure of a body's rotational inertia is called **moment of inertia** and it is represented by I. The moment of inertia of a body is a function of the mass of the body, the distribution of that mass and the position of the axis of rotation.

Consider a particle of mass m situated at a distance r from the axis as shown in the figure. Its moment of inertia I is defined as

$$I = mr^2 \qquad \ldots(4)$$

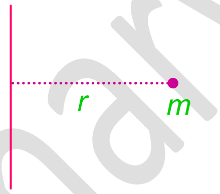


Fig. (2)

If a system of particles is made of number of particles of masses m_1 , m_2, m_3, ... m_n at distances r_1, r_2, r_3 ,... r_n from the axis of rotation, its moment of inertia is defined as

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_nr_n^2$$

$$= \sum_{i=1}^{i=n} m_i r_i^2 \qquad \ldots(5)$$

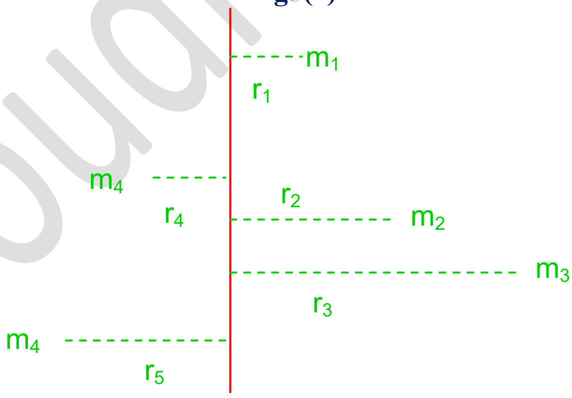


Fig. (3)

2.2 MOMENT OF INERTIA OF CONTINUOUS BODY

For calculating moment of inertia of a continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice depends on symmetry of body. Consider an element of the body at a distance r from the axis of rotation. The moment of inertia of this element about the axis we define as $(dm)\ r^2$ and the discrete sum over particles becomes integral over the body:

$$I = \int (dm) r^2 \qquad \ldots(6)$$

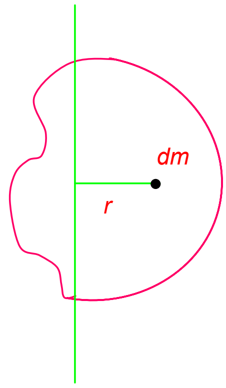


Fig. (4)

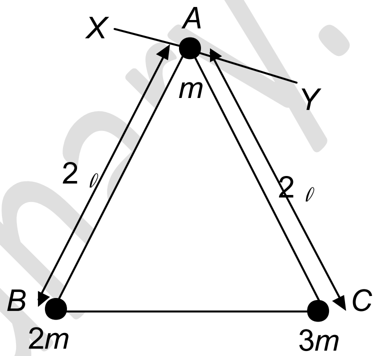
Illustration 4

Question: **Three light rods, each of length 2ℓ(ℓ = 1m), are joined together to form a triangle. Three particles *A*, *B*, *C* of masses *m* (*m* =1kg), $2m$, $3m$ are fixed to the vertices of the triangle. Find the moment of inertia of the resulting body about**

(a) an axis through A perpendicular to the plane *ABC*,

(b) an axis passing through *A* and the midpoint of *BC*.

Solution: (a) *B* is distant 2ℓ from the axis *XY*



So the moment of inertia of *B* (I_B) about *XY* is $2m(2\ell)^2$

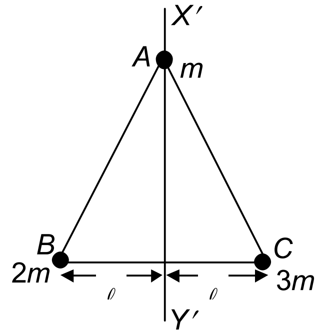
Similarly I_C about *XY* is $3m(2\ell)^2$

and I_A about *XY* is $m(0)^2$

Therefore the amount of inertia of the body about *XY* is

$2m(2\ell)^2 + 3m(2\ell)^2 + m(0)^2 = 20\ m\ell^2 =$ **20 kg m²**

(b) I_A about $X'Y' = m(0)^2$



I_B about $X'Y' = 2m(\ell)^2$

I_C about $X'Y' = 3m(\ell)^2$

Therefore the moment of inertia of the body about $X'Y'$ is

$m(0)^2 + 2m(\ell)^2 + 3m(\ell)^2 = 5\ m\ell^2 =$ **5 kg m²**

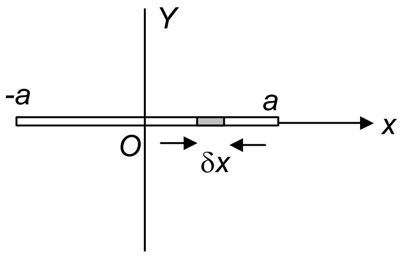
Illustration 5

Question: **A rod is of mass *M* = 3kg and length 2*a* (*a* = 2m). Find moment of inertia about an axis**

(a) through the centre of the rod and perpendicular to the rod,

(b) parallel to the rod and distant *d* = 2m from it.

Solution: Let the rod be divided into elements of length *dx*, each element being approximately a particle.



(a) For a typical element,

mass $= \frac{M}{2a}dx$

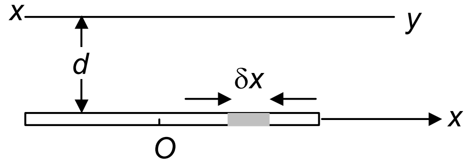
moment of inertia about $YY' = \left(\frac{M}{2a}dx\right)x^2$

Therefore $I_{YY'}$, the moment of inertia of the rod about YY' is given by

$$I_{YY'} = \frac{M}{2a}\int_{-a}^{a} x^2 dx$$

$$= \frac{1}{3}Ma^2 = \textbf{4 kg m}^2$$

(c) In this case every element of the rod is the same distance, d, from the axis XY. The moment of inertia of an element about $XY = \left(\frac{M}{2a}dx\right)\left(d^2\right)$



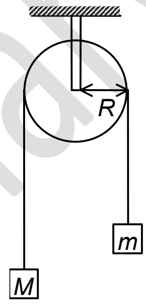
Therefore the moment of inertia of the rod about $I_{XY} = \int_0^{2a}\left(\frac{M}{2a}\right)dx\left(d^2\right) = Md^2$

$= \mathbf{12\ kg\ m^2}$

Illustration 6

Question: **The pulley shown in Figure has moment of inertia $I = 8$ kg m^2 about its axis and radius $R = 1$ m. Find the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley. ($M = 10$ kg, $m = 2$ kg)**

Solution: Suppose the tension in the left string is T_1 and that in the right string is T_2. Suppose the block of mass M goes down with an acceleration a and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim.



The angular acceleration of the wheel $\alpha = \frac{a}{R}$. The equations of motion for the mass M, the mass m and the pulley are as follows;

$Mg - T_1 = Ma$ .... (i)

$T_2 - mg = ma$ .... (ii)

$T_1R - T_2R = I\alpha = \frac{Ia}{R}$. ... (iii)

Substituting for T_1 and T_2 from equations (i) and (ii) in equation (iii)

$$[M(g-a)-m(g+a)]R = \frac{Ia}{R}$$

Solving, we get

$$a = \frac{(M-m)gR^2}{I+(M+m)R^2} = \mathbf{4\ m/s^2}$$

2.3 RADIUS OF GYRATION

The moment of inertia of any rigid body about a specified axis can be expressed in the form MK^2 where M is the mass of the body and K is a length. This is the same as the moment of inertia of a particle of mass M distant K from the axis, and K is called the radius of gyration of the body about that axis.

i.e., $I = MK^2$ …(7)

Consider, for example, a uniform rod of mass M and length 2ℓ rotating about an axis through it centre and perpendicular to the rod. If I is the moment of inertia of the rod about this axis then

$I = \frac{M\ell^2}{3} = M\left(\frac{\ell}{\sqrt{3}}\right)^2$. So radius of Gyration of the rod about axis through its centre of and perpendicular to the rod $= \frac{\ell}{\sqrt{3}}$

Many times we are tempted to replace a rotating rigid body by a particle of equal mass at the centre of gravity, but the above example shows that this does not give the correct result for the moment of inertia.

2.4 CHANGE OF AXIS

Up to this point we have usually calculated the moment of inertia of a body about an axis which passes through its centre of mass. If the moment of inertia about a different axis is required, we do not always have to go back to first principles. In some cases the following theorems provide an easy way to find the required moment of inertia.

(i) The parallel Axis Theorem

If the moment of inertia of a uniform body of mass M about an axis through G, its centre of mass, is I_G, and I_A is the moment of inertia about a parallel axis through a point A, then

$$I_A = I_G + Md^2 \qquad \text{...(8)}$$

Where d is the distance between the parallel axes.

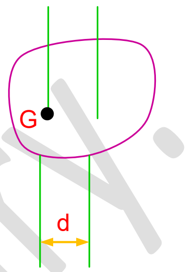


Fig. (5)

(ii) The perpendicular Axes Theorem

If a plane body has moments of inertia I_{OX} and I_{OY} about two perpendicular axes, OX and OY, in the plane of the body then its moment of inertia about an axis OZ, perpendicular to the plane, is $I_{OX} + I_{OY}$.

i.e., $$I_{OZ} = I_{OX} + I_{OY} \qquad \text{...(9)}$$

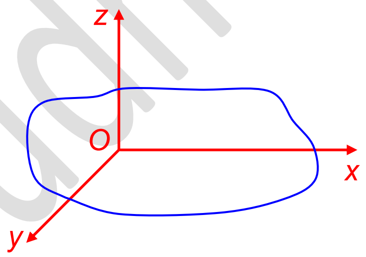


Fig. (6)

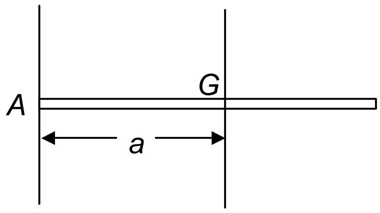
Note three axes under consideration must be mutually perpendicular and concurrent, although they need to pass through the centre of mass of the body.

This theorem cannot be applied to three-dimensional bodies.

Illustration 7

Question: **Use the parallel axis theorem to find the moment of inertia of a uniform rod of mass $M = 3$kg and length $2a$ ($a = 1$m), about a perpendicular axis through one end.**

Solution: The moment of inertia, I_G, about an axis through G and perpendicular to the rod is $\frac{1}{3}Ma^2$



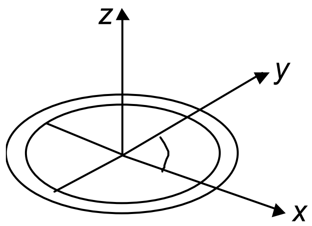
The axis through the end A is a parallel axis, therefore

$$I_A = I_G + Ma^2 = \frac{1}{3}Ma^2 + Ma^2$$

$$= \frac{4}{3}Ma^2 = \mathbf{4\ kg\ m^2}$$

Illustration 8

Question: **Find the moment of inertia, about a diameter, of a uniform ring of mass $M = 8$kg and radius $a = 1$m.**



Solution: We know that the moment of inertia, I_{OZ}, of the ring about OZ is Ma^2. We also know that, from symmetry, the moment of inertia about any one diameter is the same as that about any other diameter,

i.e., $I_{OX} = I_{OY}$

Using the perpendicular axes theorem gives

$$I_{OZ} = I_{OX} + I_{OY}$$

$$\Rightarrow \quad Ma^2 = 2I_{OX} = 2I_{OY}$$

The moment of inertia of the ring about any diameter $= \frac{1}{2}Ma^2$ = **4 kg m²**

2.5 MOMENT OF INERTIA OF COMPOUND BODIES

Consider two bodies A and B, rigidly joined together. The moment of inertia of this compound body, about an axis XY, is required.

If I_A is the moment of inertia of body A about XY.

I_B is the moment of inertia of body B about XY

Then, Moment of Inertia of compound body

$I = I_A + I_B$

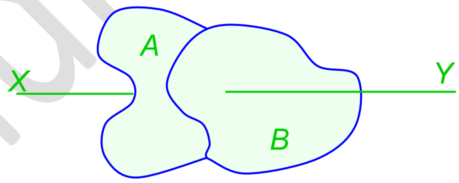


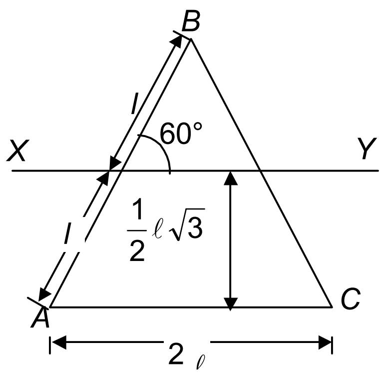
Fig. (7)

Extending this argument to cover any number of bodies rigidly joined together, we see that the moment of inertia of the compound body, about a specified axis, is the sum of the moments of inertia of the separate parts of the body about the same axis.

Illustration 13

Question: **Three uniform rods, each of length 2ℓ (ℓ = 1m) and mass M = 8 kg are rigidly joined at their ends to form a triangular framework. Find the moment of inertia of the framework about an axis passing through the midpoints of two of its sides.**

Solution: The rod AB is rotating about an axis through its midpoint and inclined to AB at 60°, therefore



For rod AB, $I_{XY} = \frac{1}{3}M\ell^2 \sin^2 60° = \frac{1}{4}M\ell^2$

Similarly for rod BC $I_{XY} = \frac{1}{4}M\ell^2$

The rod AC is rotating about an axis parallel to AC and distant $\frac{1}{2}\ell\sqrt{3}$ from AB, therefore

For rod AC, $I_{XY} = M(\frac{1}{2}\ell\sqrt{3})^2 = \frac{3}{4}M\ell^2$

Hence for the whole framework

$$I_{XY} = \frac{1}{4}M\ell^2 + \frac{1}{4}M\ell^2 + \frac{3}{4}M\ell^2 = \frac{5}{4}M\ell^2 = \mathbf{10\ kg\ m^2}$$

3 COMBINED ROTATIONAL & TRANSNATIONAL MOTION OF A RIGID BODY: ROLLING MOTION

We already learnt about translation motion caused by a force and rotation about a fixed axis caused by a torque. Now we are going to discuss a motion in which body undergoes translation as well as rotation. Rolling is an example of such motion.

Rolling motion can be considered as combination of rotational and translational motion. For the analysis of rolling motion we deal translation separately and rotation separately and then we combine the result to analyse the overall motion.

Consider a uniform disc rolling on a horizontal surface. Velocity of its center of mass is v and its angular speed is ω as shown:

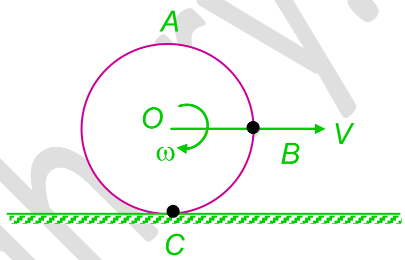


Fig. (8)

A, B and *C* are three points on the disc. Due to the translational motion each point *A, B* and *C* will move with center of mass in horizontal direction with velocity v. Due to pure rotational motion each point will have tangential velocity ωR, R is radius of disc. When the two motions are combined, resultant velocities of different points are given by

$V_A = V + \omega R$

$V_B = \sqrt{V^2 + \omega^2 R^2}$

$V_C = V - \omega R$

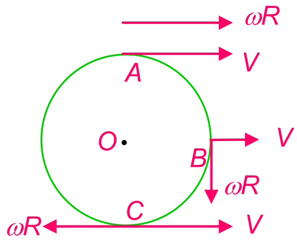


Fig. (9)

Similarly, if disc rolls with angular acceleration α and its center of mass moves with acceleration 'a' different points will have accelerations given by (for $\omega = 0$)

$a_A = a + \alpha R$

$a_B = \sqrt{a^2 + \alpha^2 R^2}$

$a_C = a - \alpha R$

To write equations of motion for rolling motion, we can apply $\vec{F}_{ext} = M\vec{a}_{CM}$ for translation motion and $\tau = I\alpha$ about axis passing through center of mass of body.

Rolling motion is possible in two ways – rolling without slipping and rolling with slipping. There is no relative motion at contact in case of rolling without slipping, while in case of rolling with slipping, relative motion takes place between contact points.

In the taken example, if rolling is without slipping we will have

$V_c = 0 \Rightarrow V = \omega R$

and, $a_c = 0 \Rightarrow a = \alpha R$

If rolling is with slipping, $V_c \neq 0$ and $a_c \neq 0$.

One more important distinction between these two kinds of rolling motion is that in case of rolling with slipping, the frictional force is a known force of magnitude μN, while in case of rolling without slipping, frictional force is of unknown magnitude. It may take any value between zero and μN.

3.1 KINETIC ENERGY OF A ROLLING BODY

If a body of mass M is rolling on a plane such that velocity of its centre of mass is V and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 \qquad \ldots(10)$$

I is moment of inertia of body about axis passing through centre of mass.

In case of rolling without slipping,

$$KE = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\because V = \omega R]$$

$$= \frac{1}{2}\left[MR^2 + I\right]\omega^2$$

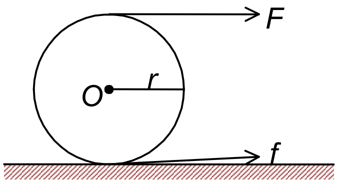
$$= \frac{1}{2}I_c\omega^2$$

I_c is moment of inertia of the body about the axis passing through point of contact.

Illustration 10

Question: **A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping find the acceleration of the centre of sphere. ($F/m = 7$)**

Solution: Suppose that the static friction (f) on the sphere acts towards right. Let r be the radius of sphere and a the linear acceleration of centre of sphere. The angular acceleration about the centre is $\alpha = \frac{a}{r}$ as there is no slipping. For the linear motion of centre,



$$F + f = ma \qquad \ldots(i)$$

For rotational motion about centre

$$Fr - fr = I\alpha$$

$$I = \frac{2}{5}mr^2 \qquad \alpha = \frac{a}{r}$$

$$\therefore \qquad r(F - f) = \frac{2}{5}mr^2 \cdot \frac{a}{r}$$

$$F - f = \frac{2}{5}ma \qquad \ldots(ii)$$

Adding (1) and (2)

$$2F = \frac{7}{5}ma \qquad a = \frac{10F}{7m} = \mathbf{10\ m/s^2}$$

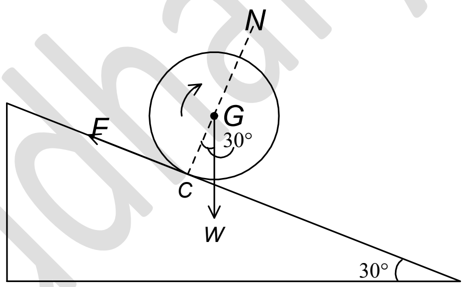
Illustration 11

Question: **A sphere of radius *r* and mass *m* is released at rest on the incline whose inclination with horizontal is 30°. It rolls without slipping. Determine**

(a) the velocity of the centre *G* of the sphere after the sphere has rolled down 7 m,

(b) the velocity of *G* if the sphere were to move 5 m down the same incline when there is no friction.

Solution: The external forces W, N and F form a system equivalent to the system of effective forces represented by the vector $m\vec{a}$ and couple $I\alpha$.



Since the sphere rolls without slipping, we have $a = r\,\alpha$ where a is linear acceleration and α is angular acceleration and r the radius of sphere.

From the Figure, we see

$$W \sin\theta \cdot r = I_C\,\alpha$$

$$\therefore \quad W \sin\theta \cdot r = \frac{7}{5}mr^2\alpha$$

But $\quad W = mg$

$$mg \sin\theta \cdot r = \frac{7}{5}mr^2\alpha$$

But $\alpha = \dfrac{a}{r}$

$$mg \sin\theta \cdot r = \frac{7}{5}mr^2 \cdot \frac{a}{r}$$

$$\therefore \quad a = \frac{5g\sin\theta}{7} = \frac{5\times 9.8\times \sin 30^\circ}{7} = 3.5 \text{ m/s}^2.$$

If F is the force of friction between the sphere and the inclined plane, considering the linear motion,

$$mg\sin\theta - F = ma$$

$$mg\sin\theta - F = m\cdot\frac{5g\sin\theta}{7}$$

But $\quad F = \mu N$

And $\quad N = mg\cos\theta$

$$\therefore \quad mg\sin\theta - \mu\, mg\cos\theta = m\cdot\frac{5g\sin\theta}{7}$$

$\mu\, mg \cos\theta = mg\sin\theta - \frac{5}{7} mg\sin\theta$

$\mu\, mg \cos\theta = \frac{2}{7} mg\sin\theta$

$\mu = \frac{2}{7}\tan\theta = \frac{2}{7}\tan 30^\circ = \frac{2}{7}\times\frac{1}{\sqrt{3}}$

$\mu = 0.165$

(a) To calculate the velocity of the centre of the sphere after it has moved a distance 7 m.

Initial velocity = 0

$a = 3.5$ m/s^2

Distance = 7 m

Using $v^2 - u^2 = 2aS$

$v^2 = 0 + 2 \times 3.5 \times 7$

$v^2 = 49$

$v =$ **7 m/s**

(b) To find the velocity of sliding sphere (in the absence of friction)

$u = 0$; $S = 5$ m

$a = g \sin 30^\circ = \frac{g}{2} = 4.9$ m/s^2

$v^2 = 0 + 2 \times 4.9 \times 5 = 49$

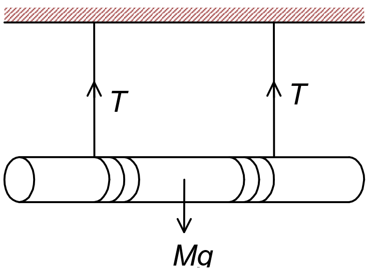
$v =$ **7 m/s**

Illustration 12

Question: **A cylinder of mass $M = 6$ kg is suspended through two strings wrapped around it as shown in Figure. Find the tension in the string and the speed of the cylinder as it falls through a distance $h = 30$ m.**

Solution:

The portion of the strings between ceiling and cylinder are at rest. Hence the points of the cylinder where the strings leave it are at rest also. The cylinder is thus rolling without slipping on the strings. Suppose the centre of cylinder falls with an acceleration a. The angular acceleration of cylinder



$$\alpha = \frac{a}{R} \qquad \ldots(i)$$

as the cylinder does not slip over the strings. The equation of motion for the centre of mass of cylinder is

$$Mg - 2T = Ma$$

and for the motion about the centre of mass it is

$$2T\cdot R = \left(\frac{MR^2}{2}\right)\alpha \text{, where } I = \frac{MR^2}{2}$$

$$2TR = \frac{MR^2}{2}\cdot\frac{a}{R}$$

$$2T = \frac{Ma}{2} \qquad \text{... (ii)}$$

From (i) and (ii) on adding

$$Mg = \frac{Ma}{2} + Ma \quad \frac{3a}{2} = g$$

$$a = \frac{2g}{3}$$

$$\therefore \quad 2T = \frac{M}{2}\cdot\frac{2g}{3}$$

$T = Mg/6 =$ **10 N**

As the centre of cylinder starts moving from rest, the velocity after it has fallen a height h is given by

$$v^2 = 2\left[\frac{2g}{3}\right]h \quad \text{or} \quad v = \sqrt{\frac{4gh}{3}} = \mathbf{20\ m/s}$$

3.2 ANGULAR MOMENTUM OF ROLLING BODY

Angular momentum of a rolling body having angular velocity ω and velocity of center of mass V is given by

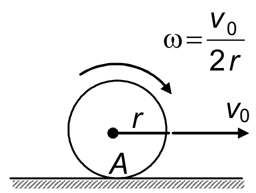
$$L = MVr + I_{CM}\,\omega$$

Here r is perpendicular distance of line of motion of mass from the point about which angular momentum is to be calculated.

Note that angular momentum is a vector quantity so while adding the direction of angular momentum should be given proper attention.

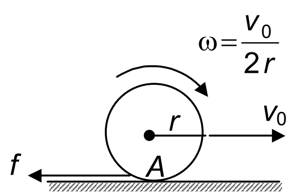
Illustration 13

Question: **A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 = 7 m/s and rotational velocity $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.**



Solution:

Let us consider the torque about the initial point of contact A. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about A is zero. Hence the angular momentum about A is conserved. Initial angular momentum is,



$$L = L_{cm} + Mrv_0 = I_{cm}\,\omega + Mrv_0$$

$$= \left(\frac{2}{5}Mr^2\right)\left(\frac{v_0}{2r}\right) + Mrv_0 = \frac{6}{5}Mrv_0$$

Suppose the translational velocity of the sphere, after it starts rolling, is v. The angular velocity is v/r. The angular momentum about A is,

$$L = L_{cm} + Mrv$$

$$= \left(\frac{2}{5}Mr^2\right)\left(\frac{v}{r}\right) + mrv = \frac{7}{5}Mrv.$$

Thus, $\frac{6}{5}Mrv_0 = \frac{7}{5}Mrv$

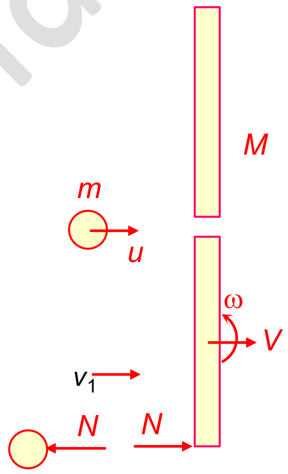
or $\quad v = \frac{6}{7} v_0 = \mathbf{6\ m/s}$.

☞ *A body rolling without slipping on a fixed surface can also be analysed as pure rotation about the axis passing through the point of contact.*

4 COLLISION OF POINT MASSES WITH RIGID BODIES (ECCENTRIC COLLISION)

In the lesson, Impulse and momentum, we discussed about central impact in which the line of collision coincided with the line joining the center of mass of colliding bodies. Now we are going to discuss the collision in which the line of collision and line joining center of mass are different, i.e., Eccentric Collision.

Consider a uniform rod of mass M and length L resting on a frictionless surface. A small disc of mass m hits the rod perpendicular to its length near its end as shown in figure. The speed of disc at the time of collision is u. Let e be coefficient of restitution for the collision.



At the time of collision, forces between the rod and disc is as shown in figure. These forces on disc will cause change in velocity of disc. Let it become v_1. Force on rod will provide translational velocity v to C.M. of rod and on angular speed ω to the rod. Let us find these unknown velocities v_1, ω and v.

Taking the rod and disc as a system, $\sum \vec{F} = 0$, we can apply conservation of linear momentums to get equation :

$$mu = mv_1 + MV \qquad \ldots \text{(i)}$$

As the forces at the time of collision are equal, opposite and collinear, of these forces torque about CM is zero so we can apply conservation of angular momentum about cm,

$$Mu\frac{\ell}{2} = mv_1\frac{\ell}{2} + \frac{M\ell^2}{12}\omega \qquad \ldots \text{(ii)}$$

From the law of restitution we can write,

$$\left(V + \omega\frac{\ell}{2}\right) - V_1 = e(u - 0) \qquad \ldots \text{(iii)}$$

Solving these three equations we can calculate V_1, ω and v

Here in the taken situation the rod is free to translate and rotate. If the rod were given to rotate about a fixed axis then we would not be able to apply conservation of linear momentum and in such case two unknowns can be calculated using conservation of angular momentum about the axis of rotation and law of restitution.

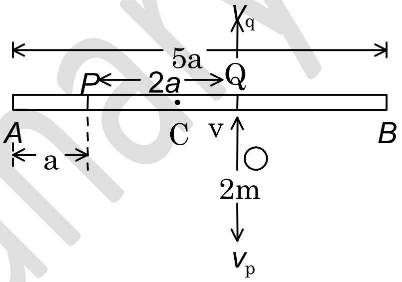
Illustration 14

Question: **A uniform rod *AB* of mass *m* and length 5*a* is free to rotate on a smooth horizontal table about a pivot through *P*, a point on *AB* such that *AP* = *a* = 1m. A particle of mass 2*m* moving on the table strikes *AB* perpendicularly at the point 2a from *P* with speed *v* = 37 m/s, the rod being at rest. If the coefficient of restitution between them is $\frac{1}{4}$, find their speeds immediately after impact.**

Solution: Let the point of impact be Q so that $PQ = 2a$

Let P be the point of pivot so that $AP = a$

Let the velocities of Q and the particle after impact be v_q and v_p respectively. We can apply three principles of motion



1. Conservation of linear momentum
2. Conservation of angular momentum
3. Newton's law of restitution for collision

However the law of conservation of linear momentum will involve the unknown impulsive reaction at P. Hence we use the latter two principles only.

By the law of conservation of angular momentum, the effective impulse on the rod at Q is equal to the change in angular momentum of the particle and so

$$2a\,(2mv + 2mv_p) = I_p\omega \qquad \dots \text{(I)}$$

where I_p is the moment of inertia of AB about P.

$$I_p = \frac{1}{3}m\left(\frac{5a}{2}\right)^2 + m\left(\frac{3a}{2}\right)^2 = \frac{13ma^2}{3}$$

$$\therefore \quad 4ma\,(v + v_p) = \frac{13ma^2}{3}\omega$$

$$12\,(v + v_p) = 13a\omega \qquad \dots \text{(ii)}$$

By Newton's law of restitution

$$v_p + v_q = \frac{v}{4} \qquad \dots \text{(iii)}$$

The angular velocity ω of the rod is such that

$$v_q = 2a\omega \qquad \dots \text{(iv)}$$

Substituting for v_p from (iii) in (ii)

$$12\left(v + \frac{v}{4} - v_q\right) = 13a\omega \quad 12\left(\frac{5v}{4} - 2a\omega\right) = 13a\omega$$

$$15v - 24a\omega = 13a\omega$$

$$\therefore \quad \omega = \frac{15v}{37a} = 15 \text{ rad/s}$$

Substituting back in (iii)

$$v_p = \frac{v}{4} - 2a\frac{15v}{37a} = \frac{v}{4} - \frac{30v}{37} = -\frac{83v}{148} = -2075 \text{ cm/s}$$

Thus the angular speed of the rod is **15 rad/s** and the speed of the particle is **2075 cm/s** after impact.

PROFICIENCY TEST

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. Can the centre of mass of a body lie outside the body? If so give examples.

2. Three balls are thrown into air simultaneously. What is the acceleration of their centre of mass while they are in motion?

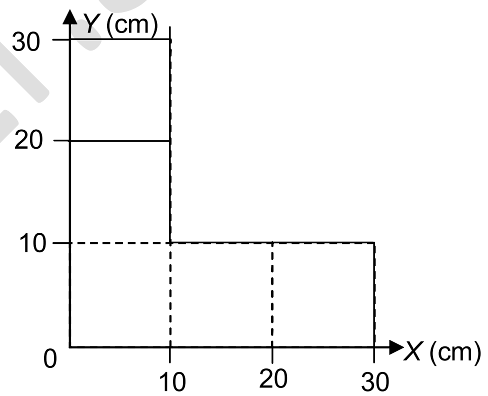
3. As a ball falls towards the earth, the momentum of the ball increases. Reconcile this fact with the law of conservation of momentum.

4. A bomb, initially at rest, explodes into several pieces.
(a) Is linear momentum constant?
(b) Is kinetic energy constant? Explain.

5. The mass of the moon is about 0.013 times the mass of earth and the distance from the centre of the moon to the centre of earth is about 60 times the radius of earth. How far is the centre of mass of earth-moon system from the centre of earth?

6. A 2.0 kg particle has a velocity $(2.0\vec{i} - 4.0\vec{j})$ m/s, and a 3.0 kg particle has a velocity $(2.0\vec{i} + 6.0\vec{j})$ m/s. Find
(a) the velocity of the centre of mass and
(b) the total momentum of the system.

7. A uniform piece of sheet is shaped as shown in the figure. Compute x and y co-ordinates of centre of mass of the piece.



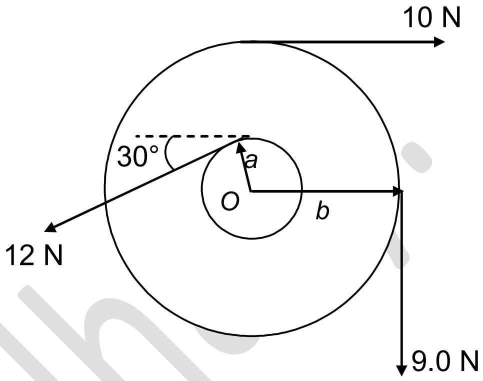
8. A 2.0 kg particle has a velocity of $\vec{v}_1 = (2.0\vec{i} - 10t\vec{j})$ m/s, where t is in seconds. A 3.0 kg particle moves with a constant velocity of $\vec{v}_2 = 7.0\vec{i}$ m/s. At $t = 0.50$ s, find (a) the velocity of the centre of mass, (b) the acceleration of the centre of mass, and (c) the total momentum of the system.

9. What is magnitude of the angular acceleration α of the second's hand?

10. Suppose that only two external forces act on a rigid body at rest, the two are equal in magnitude and opposite in direction. Can the body translate? Can the body rotate?

11. Explain why changing the axis of rotation of an object changes its moment of inertia.

12. Find the net torque on the wheel in figure about the axle through O if a = 10 cm and b = 25 cm.



13. Suppose that only two external forces act on a rigid body, and the two forces are equal in magnitude but opposite in direction. Under what conditions does the body rotate?

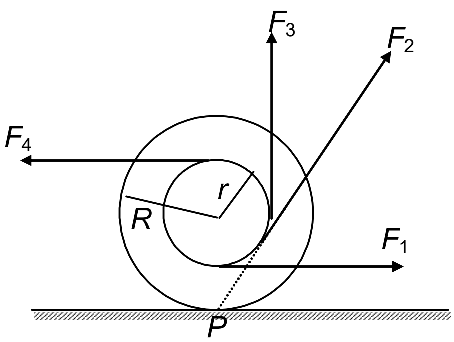
14. A turntable rotates at a constant rate of 60π rev/min. What is its angular speed in radians per second? What is the magnitude of its angular acceleration? (π^2 = 10)

15. Find the radius of gyration of (a) a solid disk of radius $R = \sqrt{2}$ m, (b) a uniform rod of length $L = 2\sqrt{12}$ m, and (c) a solid sphere of radius $R = \sqrt{\frac{5}{2}}$ m, all three rotating about a central axis.

16. Use the parallel-axis theorem to find the moments of inertia of (a) a solid cylinder about an axis parallel to its axis and passing through the surface of the cylinder and (b) a solid sphere about an axis tangent to its surface. (M = 2kg, $R = \sqrt{5}$ m)

17. The center of mass of a pitched baseball of radius $R = \sqrt{5}$ m moves at a speed v = 1 m/s. The ball spins about an axis through its center of mass with an angular speed ω = 1 rad/s. Calculate the ratio of the rotational energy to the translational kinetic energy. Treat the ball as a uniform sphere.

18. A spool of wire rests on a horizontal surface as in figure and when pulled, does not slip at the contact point P. The spool is pulled in the directions indicated by the vectors F_1, F_2, F_3, and F_4. For each force, determine the direction the spool rolls. Note that the line of action of F_2 passes through P.



19. Three objects of uniform density–a solid sphere, a solid cylinder, and a hollow cylinder–are placed at the top of a rough incline. If they all are released from rest at the same elevation and roll without slipping, which reaches the bottom first? Which reaches last?

20. Repeat the previous question if the surface is frictionless.

ANSWERS TO PROFICIENCY TEST

5. 4930 km, since the radius of earth is 6400 km.

6. (a) $2\hat{i}+2\hat{j}$ m/s (b) $10\hat{i}+10\hat{j}$ kg-m/s

7. 11cm; 11cm

8. (a) $5\hat{i}-2\hat{j}$ m/s (b) $-4\hat{j}$ m/s^2 (c) $25\hat{i}-10\hat{j}$ kg-m/s

9. zero

12. 355 Nm

14. 20 rad s^{-1}-, zero

15. 1m, 2m, 1m

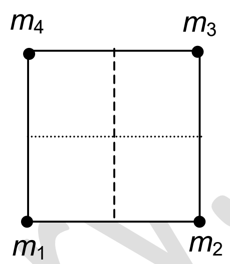
16. 15 kg m^2, 14 kg m^2

17. 2

SOLVED OBJECTIVE EXAMPLES

Example 1:

Four particles of masses $m_1 = 2m$, $m_2 = 4m$, $m_3 = m$ and m_4 are placed at four corners of a square. What should be the value of m_4 so that the centre of mass of all the four particles are exactly at the centre of the square?



(a) 2 m **(b) 8 m**
(c) 6 m **(d) none of these**

Solution:

Unless $m_1 = m_3$ the COM of all the four particles can never be at the centre of the square.

$\therefore$ **(d)**

Example 2:

Two particles of equal mass have velocities $\vec{v}_1 = 2\hat{i}$ m/s and $\vec{v}_2 = 2\hat{j}$ m/s. First particle has an acceleration $\vec{a}_1 = (3\hat{i} + 3\hat{j})$ m/s² while the acceleration of the other particle is zero. The center of mass of the two particles moves on a

(a) circle **(b) parabola** **(c) straight line** **(d) ellipse**

Solution:

$$\vec{v}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$= \frac{\vec{v}_1 + \vec{v}_2}{2} \quad (m_1 = m_2)$$

$$= (\hat{i} + \hat{j}) \text{ m/s}$$

Similarly, $\vec{a}_{COM} = \dfrac{\vec{a}_1 + \vec{a}_2}{2} = \dfrac{3}{2}(\hat{i} + \hat{j})$ m/s²

Since $\vec{v}_{COM}$ is parallel to $\vec{a}_{COM}$ the path will be a **straight line**.

$\therefore$ **(c)**

Example 3:

A rope thrown over a pulley has a ladder with a man of mass m on one of its ends and a counterbalancing mass M on its other end. The man climbs with a velocity v_r relative to ladder. Ignoring the masses of the pulley and the rope as well as the friction of the pulley axis, the velocity of the centre of mass of this system is

(a) $\dfrac{m}{M}v_r$ (b) $\dfrac{m}{2M}v_r$ (c) $\dfrac{M}{m}v_r$ (d) $\dfrac{2M}{m}v_r$

Solution:

The rope tension is the same on the left and right hand side at every instant, and, consequently, momentum of both sides are equal

$\therefore \quad Mv = (M - m)(-v) + m(v_r - v)$

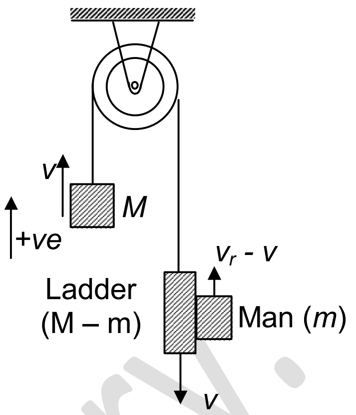
or $\quad v = \dfrac{m}{2M} v_r$

Momentum of the centre of mass is

$P = P_1 + P_2$

or $\quad v_{COM} = v = \dfrac{m}{2M} v_r$

$\therefore$ **(b)**



Example 4:

A rigid body rotates about a fixed axis with variable angular velocity equal to ($a - bt$) at time t where a and b are constants. Find the angle through which it rotates before it comes to rest.

(a) $\dfrac{a^2}{b}$ (b) $\dfrac{a^2}{2b}$ (c) $\dfrac{a^2}{4b}$ (d) $\dfrac{a^2}{2b^2}$

Solution:

$\omega = a - bt$

At time $t = 0$, $\omega = \omega_0 = a$

$\alpha = \dfrac{d\omega}{dt} = -b$

$\omega^2 = \omega_o^2 + 2\alpha\cdot\theta$

If ω is zero

$0 = \omega_o^2 + 2\alpha\theta$

$\theta = -\dfrac{\omega_o^2}{2\alpha} = \dfrac{a^2}{2b}$

$\therefore$ **(b)**

Example 5:

A wheel whose moment of inertia is 2 kg m^2 has an initial angular velocity of 50 rad/s. A constant torque of 10 Nm acts on the wheel. The time in which the wheel is accelerated to 80 rad/s is

(a) 12 s (b) 3 s (c) 6 s (d) 9 s

Solution:

Initial angular velocity = 50 rad/s
Final angular velocity = 80 rad/s
Torque = 10 N-m
Moment of Inertia = 2 kg m^2
Angular acceleration α is given by

$\tau = I\alpha$

$\alpha = \dfrac{\tau}{I} = \dfrac{10}{2} = 5$ rad/s^2

Hence if t is the time $\quad 5t = 80 - 50 = 30$

t = **6 seconds**

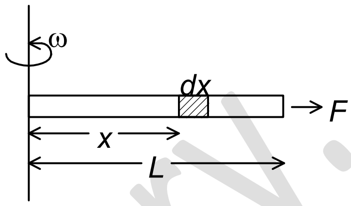
$\therefore$ **(c)**

Example 6:

A tube of length L is completely filled with an incompressible liquid of mass M and closed at both ends. The tube is rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the other end will be

(a) $\frac{M\omega^2 L}{2}$ (b) $M\omega^2 L$ (c) $\frac{M\omega^2 L}{4}$ (d) $\frac{M\omega^2 L^3}{2}$

Solution:



Consider a small element of the liquid of length dx at a distance x from the axis of rotation. Its mass is $\frac{M}{L}\,dx$.

Centrifugal force $= \frac{Mdx}{L}\omega^2 x$

Total force $= \frac{M}{L}\omega^2 \int_0^L x\,dx$

$= \frac{M}{L}\omega^2 \frac{L^2}{2}$

$= \frac{M\omega^2 L}{2}$

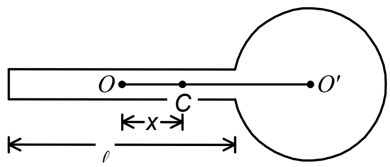
∴ (a)

Example 7:

A table tennis racket has a flat rectangular handle of mass m and length ℓ and a flat circular disc of radius r and mass m attached to the handle. The moment of inertia of the bat about an axis perpendicular to its plane and passing through its centre of mass is

(a) $\frac{m\ell^2}{12}+\frac{mr^2}{2}$ (b) $\frac{m\ell^2}{3}+\frac{3mr^2}{2}$ (c) $2m\left(\frac{\ell}{2}+r\right)^2$ (d) $\frac{m\ell^2}{12}+\frac{mr^2}{2}+\frac{m}{8}(\ell+2r)^2$

Solution:



Let O be the centre of mass of the handle and O' that of the disc. Let x be the distance of C.M of the system C from O.

Then $mx = m\left(\frac{\ell}{2}+r-x\right)$

$x = \frac{\ell+2r}{4} = OC = O'C$

∴ required moment of inertia $= \left[\frac{m\ell^2}{12}+m\left(\frac{\ell+2r}{4}\right)^2\right]+\left[\frac{mr^2}{2}+m\left(\frac{\ell+2r}{4}\right)^2\right]$

$= \frac{m\ell^2}{12}+\frac{mr^2}{2}+\frac{m}{8}(\ell+2r)^2$

∴ (d)

Example 8:

A man of 80 kg is standing on the rim of a circular platform of mass 200 kg. The platform rotates about its axis at 12 r.p.m. The man moves from rim to centre of the platform. How will the system rotate? (The moment of inertia of man at the centre may be neglected.)

(a) at 10 r.p.m (b) at 12 r.p.m (c) at 21.6 r.p.m (d) stop rotating

Solution:

If r is the radius of the platform and M its mass,

Moment of inertia of platform about the axis $= \frac{Mr^2}{2}$.

Moment of inertia of the system with the man at the rim $= \frac{Mr^2}{2} + mr^2$

$= \frac{200r^2}{2} + 80r^2$

$= 180\ r^2$

Moment of inertia with the man at the centre is $\frac{Mr^2}{2} = 100\ r^2$

By conservation of angular momentum, $180\ r^2\ \omega_1 = 100\ r^2\ \omega_2$

$$\omega_2 = \frac{180}{100}\ \omega_1 = \frac{180 \times 12}{100} = \mathbf{21.6\ rpm}$$

$\therefore$ **(c)**

***Example* 9:**

A thin circular ring of mass *M* and radius *R* is rotating about its axis with a constant angular velocity ω. Two objects each of mass *m* are attached gently to the ring. The wheel now rotates with an angular velocity

(a) $\frac{\omega M}{M+m}$ **(b)** $\frac{\omega(M-2m)}{M+2m}$ **(c)** $\frac{\omega M}{M+2m}$ **(d)** $\frac{\omega(M+2m)}{M}$

Solution:

Moment of inertia of a circular ring about an axis passing through its centre and perpendicular to its plane is MR^2.

Initial angular momentum $= MR^2\ \omega$

After the masses have been attached, the moment of inertia $= MR^2 + 2mR^2$

If ω' is the new angular velocity, the angular momentum $= (MR^2 + 2mR^2)\ \omega'$

By the principle of conservation of angular momentum

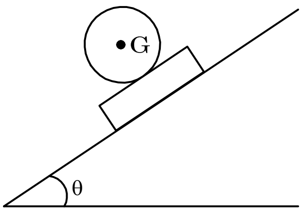
$(MR^2 + 2mR^2)\ \omega' = MR^2\ \omega$

$\therefore\ \omega' = \frac{\omega M}{M+2m}$

$\therefore$ **(c)**

***Example* 10:**

The acceleration *a* of the supporting surface (see Figure) required to keep the centre *G* of the circular cylinder in a fixed position during the motion if there is no slipping between the cylinder and the support will be



(a) $g \sin\theta$ **(b)** $\frac{(g\sin\theta)}{2}$

(c) $2g \sin\theta$ **(d)** $4g \sin\theta$

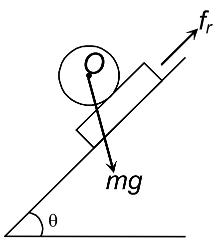
Solution:

Net force on the cylinder acting parallel to the inclined plane.

For the motion of cylinder,

$ma' = mg \sin\theta - f_r$

The centre of mass of cylinder is stationary. $a' = \frac{d^2x}{dt^2} = 0$



$\therefore\quad f_r = mg \sin\theta$

Torque on the cylinder about $O = f_r R = \frac{mR^2}{2}\alpha$

$f_r = \frac{mR\alpha}{2} = \frac{ma}{2}$

$\frac{ma}{2} = mg \sin\theta$

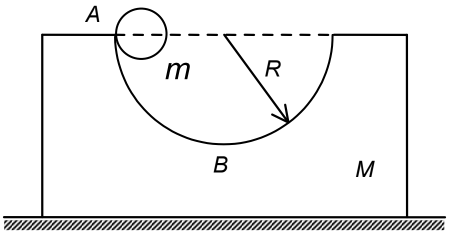
$\therefore$ $a = \mathbf{2g \sin\theta}$

$\therefore$ **(c)**

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

A block of mass M = 4 kg with a semicircular track of radius R = 5 m rests on a horizontal frictionless surface. A uniform cylinder of radius r = 1m and mass m = 6kg is released from rest at the top point A (see Figure). The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reached the bottom (point B) of the track? How fast is the block moving when the cylinder reaches the bottom of the track? ($\sqrt{2} = 1.4$)



Solution:

The horizontal component of forces acting on M-m system is zero and the centre of mass of the system cannot have any horizontal displacement.

When the cylinder is at B its displacement relative to the block in the horizontal direction is $(R - r)$. Let the consequent displacement of the block to the left be x. The displacement of the cylinder relative to the ground is $(R - r - x)$.

Since the centre of mass has no horizontal displacement

$$M \cdot x = m\,(R - r - x)$$

$$x\,(M + m) = (R - r)\,m$$

$$x = \frac{(R-r)m}{(M+m)}$$

When the cylinder is at A, the total momentum of the system in the horizontal direction is zero. If v is the velocity of the cylinder at B and V, the velocity of the block at the same instant, then

$mv + MV = 0$, by principle of conservation of momentum.

Potential energy of the system at A $= mg\,(R - r)$

Kinetic energy of the cylinder at B $= \frac{1}{2}mv^2$

The kinetic energy of the block at that instant $= \frac{1}{2}MV^2$

By principle of conservation of energy,

$$mg\,(R - r) = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

since $v = -\frac{MV}{m}$

$$mg\,(R - r) = \frac{1}{2}m\left(-\frac{MV}{m}\right)^2 + \frac{1}{2}MV^2 = \frac{V^2}{2}\left(\frac{M^2}{m} + M\right)$$

$$mg\,(R - r) = \frac{V^2}{2m}\left(M^2 + Mm\right)$$

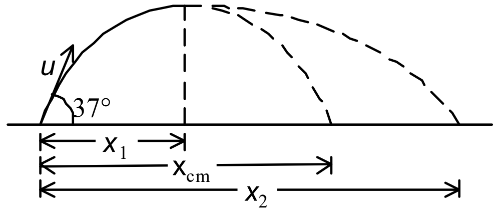
$$V^2 = \frac{2m^2 g(R-r)}{(M^2+Mm)}$$

V = **840 cm/s**

Example 2:

A projectile is fired at a speed of 100 m/s at an angle of 37° above horizontal. At the highest point the projectile breaks into two parts of mass ratio 1 : 3, the lighter coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Solution:



Refer the Figure. At the highest point, the projectile has horizontal velocity. The lighter part comes to rest. Hence the heavier part will move with increased velocity in the horizontal direction. In the vertical direction both parts have zero velocity and undergo same acceleration. Hence they will cover equal vertical displacements in a given time. Thus both will hit the ground together. As internal forces do not affect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is

$$X_m = \frac{2u^2 \sin\theta\cos\theta}{g}$$

$$= \frac{2\times(100)^2\times\frac{3}{5}\times\frac{4}{5}}{10} = 960 \text{ m}$$

where $\sin\theta=\frac{3}{5}, \cos\theta=\frac{4}{5}$ and $g = 10$ m/s^2.

The centre of mass will hit the ground at this position. As the lighter mass comes to rest after breaking it falls down vertically and hits the ground at half the range = 480 m. If the heavier block hits the ground at x_2,

$$x_{cm} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

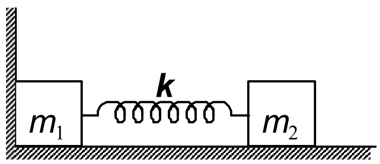
$$960 = \frac{\frac{M}{4}\times 480+\frac{3M}{4}\times x_2}{M}$$

Solving, x_2 = **1120 m**

Example 3:

Two blocks of masses m_1 = 5 kg and m_2 = 10 kg connected by a weightless spring of stiffness k = 90 N/m rest on a smooth horizontal plane. Block 2 is shifted a small distance x = 1 m to the left and released. Find the velocity of the centre of mass of the system after block 1 breaks off the wall.

Solution:



We know that the potential energy of compression

$$= \frac{1}{2}kx^2$$

When the block m_1 breaks off from the wall the spring has its unstretched length and the kinetic energy of the block m_2 is given by

$$\frac{1}{2}m_2v_2^2 = \frac{1}{2}kx^2$$

$$v_2^2 = \frac{kx^2}{m_2}$$

$$v_2 = x\sqrt{\frac{k}{m_2}}$$

For the velocity of centre of mass

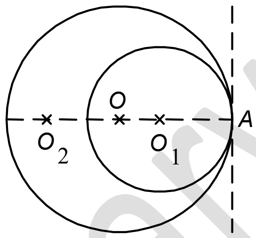
$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m_2}{m_1 + m_2} \cdot v_2 = \frac{m_2}{m_1 + m_2}\sqrt{\frac{k}{m_2}}$$

Velocity of centre of mass of system = **2m/s**

***Example* 4:**

A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Figure. Find the distance of centre of mass of the remaining portion.



Solution:

Let O be the centre of circular plate and O_1, the centre of circular portion removed from the plate. Let O_2 be the centre of mass of the remaining part.

Area of original plate = $\pi R^2 = \pi\left(\frac{56}{2}\right)^2 = 28^2\ \pi\ \text{cm}^2$

Area removed from circular part = πr^2

$$= \pi\left(\frac{42}{2}\right)^2 = (21)^2 \pi\ \text{cm}^2$$

Let σ be the mass per cm^2. Then
mass of original plate, $m = (28)^2\ \pi\sigma$
mass of the removed part, $m_1 = (21)^2\ \pi\sigma$
mass of remaining part, $m_2 = (28)^2\ \pi\sigma - (21)^1\ \pi\sigma = 343\ \pi\sigma$

Now the masses m_1 and m_2 may be supposed to be concentrated at O_1 and O_2 respectively. Their combined centre of mass is at O. Taking O as origin we have from definition of centre of mass,

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$x_1 = OO_1 = OA - O_1A = 28 - 21 = 7$ cm

$x_2 = OO_2 = ?,\ x_{cm} = 0.$

$$\therefore \quad 0 = \frac{(21)^2 \pi\sigma \times 7 + 343\pi\sigma \times x_2}{(m_1 + m_2)}$$

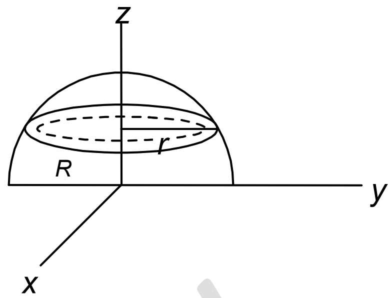
$$x_2 = -\frac{(21)^2 \pi\sigma \times 7}{343\pi\sigma} = -\frac{441 \times 7}{343} = -9 \text{ cm.}$$

This means that centre of mass of the remaining plate is at a distance **9 cm** from the centre of given circular plate opposite to the removed portion.

Example 5:

Find the z-coordinate of centre of mass of a uniform solid hemisphere of radius R = 8m and mass M with centre of sphere at origin and the flat of the hemisphere in the x, y plane.

Solution:



Let the centre of the sphere be the origin and let the flat of the hemisphere lie in the x-y plane as shown. By symmetry, x and y coordinates of centre of mass $\bar{x}=\bar{y}=0$. Consider the hemisphere divided into a series of slices parallel to x, y plane. Each slice is of thickness dz.

The slice between z and $(z + dz)$ is a disk of radius, $r=\sqrt{R^2-z^2}$.

Let ρ be the constant density of the uniform hemisphere.

Mass of the slice, $dm = (\rho\pi r^2)dz = \rho\pi(R^2 - z^2)\,dz$

The $\bar{z}$ value is obtained by $\bar{z} = \dfrac{\int_0^R z\,dm}{M} = \dfrac{\int_0^R \pi\rho(R^2z-z^3)dz}{M} = \dfrac{\pi\rho}{M}\left[\left(\dfrac{R^2z^2}{2}-\dfrac{z^4}{4}\right)\right]_{z=0}^{z=R}$

$$\Rightarrow \quad \bar{z}=\frac{\pi\rho\left(\frac{R^4}{2}-\frac{R^4}{4}\right)}{M}$$

$$\Rightarrow \quad \bar{z}=\frac{\rho\pi R^4}{4M}$$

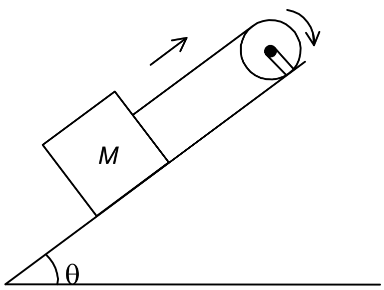
Since $2M = \rho\left(\dfrac{4}{3}\pi R^3\right)$,

we have $\bar{z}=\dfrac{(\rho\pi R^4/4)}{(\rho 2\pi R^3/3)} = \dfrac{3}{8}R = \mathbf{3\ m}$

Example 6:

A wheel of radius r = 1m and moment of inertia I = 4 kg m^2 about its axis is fixed at the top of an inclined plane of inclination θ = 30° as shown in Figure. A string is wrapped around the wheel and its free end supports a block of mass M = 2kg which can slide on the plane. Initially the wheel is rotating at speed ω = π rad/s in a direction such that the block slides up the plane. How far will the block move before stopping? ($\pi = \sqrt{10}$)

Solution:



Suppose the deceleration of the block is a. The linear deceleration of the rim of wheel is also a. The angular deceleration of wheel = a/r.

If the tension in the string is T, the equations of motion can be written as

$Mg\sin\theta - T = Ma$

$T\times r = I\alpha = I.\dfrac{a}{r}$.

Eliminating T from these equations

$$Mg\sin\theta-\frac{Ia}{r^2}=Ma$$

$$\therefore \quad a=\frac{Mgr^2\sin\theta}{I+Mr^2}$$

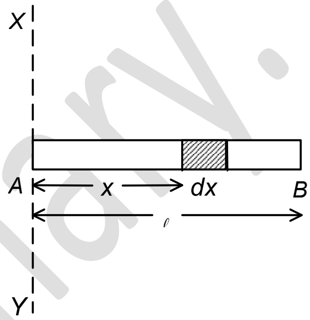
The initial velocity of block up the incline is $v = \omega r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2Mgr^2 \sin\theta} = \mathbf{3\ m}$$

Example 7:

A thin rod AB of length ℓ = 1m is such that its mass density increases uniformly from ρ at A to 4ρ at B, its total mass being M = 30 kg. Find the moment of inertia of the rod about the axis through A perpendicular to AB.

Solution:



Given that the density of the thin rod AB increases uniformly from ρ at A to 4ρ at B to find the moment of inertia of the rod AB about the axis XY through A, consider an elementary strip of the rod of length dx situated at a distance x from A.

Let the length of the rod AB be ℓ and its area of cross-section, a. The density of the rod at distance x is given by

$$\rho_x = \rho + \frac{x(4\rho - \rho)}{\ell}$$

$$= \frac{\ell\rho + x \cdot 3\rho}{\ell}$$

Mass of the elementary strip = $\rho_x\, a\, dx$

$$dm = \frac{a(\ell + 3x)\rho}{\ell} \cdot dx$$

Moment of inertia of this strip about axis XY is $x^2 \cdot dm$

$$= \frac{a(\ell + 3x)\rho}{\ell} x^2\, dx$$

Moment of inertia of the whole rod AB about XY will be $\int_0^\ell x^2\, dm$

$$= \int_0^\ell \frac{a(\ell + 3x)\rho}{\ell} \cdot x^2\, dx = \frac{a\rho}{\ell}\int_0^\ell (\ell + 3x)x^2\, dx = \frac{a\rho}{\ell}\left\{\ell\left[\frac{x^3}{3}\right]_0^\ell + 3\left[\frac{x^4}{4}\right]_0^\ell\right\}$$

$$= \frac{a\rho}{\ell}\left[\frac{\ell^4}{3} + \frac{3\ell^4}{4}\right]$$

$$\therefore \quad I = a\rho\ell^3 \cdot \frac{13}{12}$$

To express this in terms of the mass of the rod we calculate M

$$M = \int_0^\ell dm = \int_0^\ell \frac{a\rho}{\ell}(\ell + 3x)dx = \frac{a\rho}{\ell}\left\{\ell[x]_0^\ell + 3\left[\frac{x^2}{2}\right]_0^\ell\right\}$$

$$= \frac{a\rho}{\ell}\left[\ell^2 + \frac{3\ell^2}{2}\right] = \frac{5a\rho\ell^2}{2\ell} = \frac{5}{2}a\rho\ell$$

$$\therefore \quad a\rho\ell = \frac{2M}{5}$$

Substituting this value of $a\rho\ell$ in the expression for moment of inertia

we get $I = \frac{2M}{5}\left(\frac{13}{12}\right)\ell^2$ = **13 kg m²**

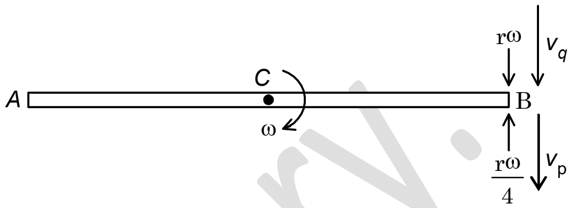
***Example* 8:**

A uniform rod of length 2*r* (*r* = 2m) and mass *m* = 4kg is rotating in a horizontal plane about a smooth fixed pivot through the centre at a steady speed of ω = 4 rad/s. A particle of mass *m* moving with speed $\frac{\omega r}{4}$ strikes the end of the rod perpendicularly, when the rod and particle were moving towards each other. The coefficient of restitution is $\frac{1}{2}$. Find the impulsive reaction at the pivot and the new speed of rod.

Solution:

Let C be the centre of the rod which is rotating about C with an angular velocity ω. Let v_p and v_q be the velocities of the particle and the end B of the rod just after impact.

Then by principal of conservation of angular momentum (taken about the pivot at C)



$\therefore \quad r\left[\frac{m(r\omega)}{4}+mv_p\right]=I(\omega-\omega')$ where ω′ is the new clockwise angular velocity of the rod after impact.

$$r\left[\frac{mr\omega}{4}+mv_p\right] = \frac{mr^2}{3}(\omega-\omega') \qquad \text{... (i)}$$

$$\frac{r\omega}{4}+v_p = \frac{r}{3}(\omega-\omega') \qquad \text{... (ii)}$$

By Newton's law of restitution $\therefore e = \frac{1}{2}$

$$v_p-v_q=\frac{\left(r\omega+\frac{r\omega}{4}\right)}{2}=\frac{5r\omega}{8} \qquad \text{... (iii)}$$

Also the velocity v_q of the rod immediately after impact is such that

$$v_q = r\,\omega' \qquad \text{... (iv)}$$

Putting the value of v_q from (iv) in (iii)

$$v_p = v_q + \frac{5r\omega}{8}$$

$$= r\,\omega' + \frac{5r\omega}{8} \qquad \text{... (v)}$$

Substituting the value of v_p in (ii) we get

$$\frac{r\omega}{4}+r\omega'+\frac{5r\omega}{8}=\frac{r}{3}(\omega-\omega') \Rightarrow \omega'+\frac{\omega'}{3} = \frac{\omega}{3}-\frac{\omega}{4}-\frac{5\omega}{8}$$

$$\frac{4\omega'}{3} = \left(\frac{8-6-15}{24}\right)\omega = -\frac{13\omega}{24}$$

Hence $\omega' = -\frac{13\omega}{24}\times\frac{3}{4} = -\frac{13\omega}{32}$ rad/s

Thus the new angular speed of the rod is $\left(-\frac{13\omega}{32}\right)$ rad/s.

The negative sign shows that it reverses its direction of motion on account of impact.

To find the impulsive reaction K at C –

If J be the impulse at end B, at the instant of impact, then by law of conservation of linear momentum

$$J - K = mv = 0$$

where v = linear velocity of the centre of mass

= 0 since the centre of mass is fixed on the pivot

$\therefore \quad K = J = \left[m\frac{r\omega}{4} - (-mv_p)\right]$

$= \frac{mr\omega}{4} + mv_p$

But from (v),

$v_p = \frac{5r\omega}{8} + r\omega' = \frac{5r\omega}{8} - r\left(\frac{13\omega}{32}\right)$

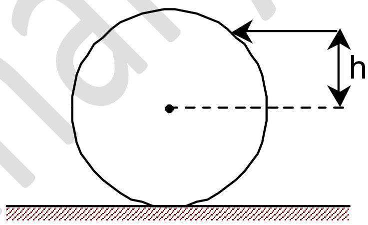
$= \frac{(20-13)r\omega}{32} = \frac{7r\omega}{32}$

$\therefore \quad K = \frac{mr\omega}{4} + \frac{7r\omega}{32} \times \text{m} =$ **15 N-s**

This gives the impulsive reaction at *C*.

Example 9:

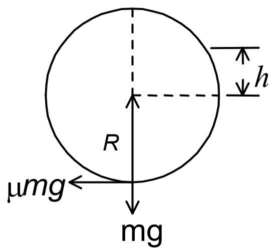
A billiard ball of radius 5m initially at rest is given a sharp impulse by a cue. The cue is held horizontally at a distance *h* above the centre line as in Figure. The ball leaves the cue with a speed v_0 and eventually acquires a speed $\frac{9}{7}$ v_0. Find the value of *h*.



Solution:

Let the initial angular velocity be ω_0. The angular momentum of the sphere is $I\omega_0 = \frac{2}{5}mR^2\omega_0$.

The moment of the impulse given = $mv_0 \cdot h$



$\therefore \quad \frac{2}{5}mR^2\omega_0 = mv_0 \cdot h$

$\omega_o = \frac{mv_o h}{\frac{2}{5}mR^2} = \frac{5v_o h}{2R^2}$

At time *t* after impact,

$v = v_0 + \mu gt$ ... (i)

$\omega = \omega_0 - \frac{\mu mgR}{\frac{2}{5}mR^2} \cdot t = \omega_o - \frac{5\mu gt}{2R}$

When $v = \omega R$ pure rolling begins.

$v_o + \mu gt = R\,\omega_o - \frac{5\mu gt}{2}$

$= \frac{5v_o h}{2R} - \frac{5}{2}\mu gt \quad \frac{7}{2}\mu gt = v_o\left(\frac{5h}{2R} - 1\right)$

$\frac{9}{7}v_0 = v_0 + \mu gt$ from (i)

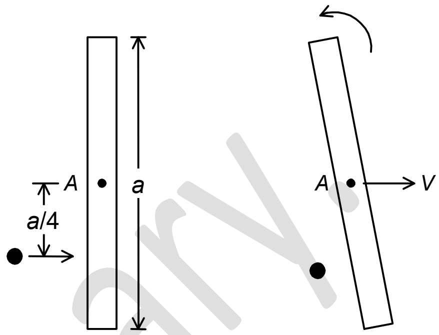
$= v_0 + \frac{2}{7}v_o\left(\frac{5h}{2R} - 1\right)$

$\frac{5h}{2R} = 2 \Rightarrow h = \frac{4R}{5} =$ **4m**

Example 10:

A uniform rod of mass M = 2kg and length a = 1m lies on a smooth horizontal plane. A particle of mass m = 1 kg moving at a speed v = 10 m/s perpendicular to the length of the rod strikes it at a distance $\frac{a}{4}$ from the centre and stops after collision. Find the velocity of the centre and the angular velocity of the rod just after collision.

Solution:



Consider the rod and particle together as the system. As there is no external resultant force; the linear momentum of the system will remain constant. Also there is no resultant external torque on the system, so the angular momentum of the system about any line will conserve.

Let V be the velocity of the centre of the rod and the angular velocity ω. By principle of conservation of linear momentum,

$mv = MV \quad V = \frac{mv}{M} = \mathbf{5\ m/s}$ ... (i)

Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB.

Initially the rod is at rest.

The angular momentum of the particle about AB, is $L = mv\left(\frac{a}{4}\right)$.

After collision the particle comes to rest. The angular momentum of rod about A is $\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{V}$

As $\quad \vec{r}_0 \| \vec{V}$, $\vec{r}_0 \times \vec{V} = 0$

Thus $\quad \vec{L} = \vec{L}_{cm}$

Hence the angular momentum of rod about AB is $L = I\omega = \frac{Ma^2}{12}\omega$.

Thus $\frac{mva}{4} = \frac{Ma^2}{12}\omega \quad \omega = \frac{3mv}{Ma} = \mathbf{15\ rad/s}$

MIND MAP

SYSTEM OF PARTICLES

1. LOCATION OF CM

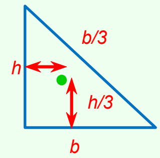
- For a system of particles

$$X_{CM}=\frac{\sum M_iX_i}{\sum M_i},\quad Y_{CM}=\frac{\sum M_iY_i}{\sum M_i},\quad Z_{CM}=\frac{\sum M_iZ_i}{\sum M_i}$$

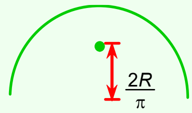
- For a continuous mass system

$$X_{CM}=\frac{\int xdm}{\int dm},\quad Y_{CM}=\frac{\int ydm}{\int dm},\quad Z_{CM}=\frac{\int zdm}{\int dm}$$

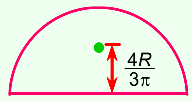
2. LOCATION OF CM FOR UNSYMMETRICAL BODIES



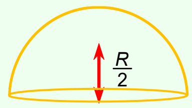
Right angled triangular lamina



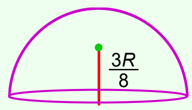
Semicircular ring



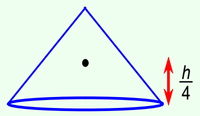
Semicircular disc



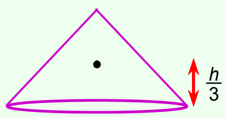
Hemispherical shell



Hemisphere



Right circular cone (solid)



Right circular cone (hollow)

4. NEWTON'S LAW

- Newton's second law of motion applicable for the system of particles:

$$\sum \vec{F}_{ext}=\left(\sum M_i\right)\vec{a}_{CM}$$

- If net external force on a system of body is zero and initially its center of mass were at rest then center of mass will remain at rest.

3. MOTION OF CM

- **Velocity of CM,**

$$\vec{V}_{cm}=\frac{\sum M_i\vec{V_i}}{\sum M_i}$$

- **Acceleration of CM,**

$$\vec{a}_{cm}=\frac{\sum M_i\vec{a_i}}{\sum M_i}$$

MIND MAP

MOTION OF RIGID BODIES

1. In case of translational motion of a rigid body, displacement, velocity and acceleration of each particle of rigid body will be same. For rigid body we apply equation of motion as

$$\vec{F}_{ext} = M\vec{a}_{CM}$$

2. Kinematic relation for the rotational motion of rigid body

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt} = \frac{\omega d\omega}{d\theta}$$

3. Equations for rigid body

$\tau = I\alpha$

$KE = \frac{1}{2} I\omega^2$

Angular momentum, $L = I\omega$

Power of torque $P = \tau\omega$

4. Moment of inertia

- For point mass system

  $I = \Sigma m_i r_i^2$

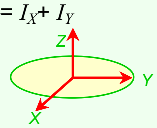
- For continuous mass system

  $I = \int dmr^2$

5. Theorems related to moment of inertia

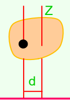
- Perpendicular axes theorem:

  $I_Z = I_X + I_Y$



- Parallel axes theorem:

  $I_Z = I_G + Md^2$



6. Rolling motion

- Equation of motion for translation

  $\vec{F}_{ext} = M\vec{a}_{cm}$

  For rotation, $\tau_{cm} = I_{cm}\alpha$

- KE of rolling body =

  $\frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\,\omega^2$

- Angular momentum of rolling body = $MVr_{cm} \pm I\omega_{cm}$

7. Important points of rolling motion

- In case of rolling without slipping frictional force may take any value between zero and $\mu_s N$.
- In case of rolling with slipping frictional force is $\mu_k N$.
- Relation between acceleration of center of mass and angular acceleration can be written in case of pure rolling motion.
- Pure rolling motion can be treated as rotation about point of contact also.

EXERCISE – I

NEET-SINGLE CHOICE CORRECT

1. A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards
(a) heavier piece
(b) lighter piece
(c) does not shift horizontally
(d) depends on the vertical velocity at the time of breaking

2. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 is the moment of inertia of the ring about an axis passing through its centre and perpendicular to its plane formed by bending this rod to the ring shape. Then
(a) $I_1 : I_2 = 1 : 1$ (b) $I_1 : I_2 = \pi^2 : 3$ (c) $I_1 : I_2 = \pi : 4$ (d) $I_1 : I_2 = 3 : 5$

3. A wheel rotates at 500 rpm on a shaft of negligible inertia (M.I.). A second identical wheel initially at rest is suddenly coupled to the same shaft. The angular speed of the resultant combination of the shaft and two wheels is
(a) 100 rpm (b) 150 rpm (c) 200 rpm (d) 250 rpm

4. A string is wrapped over the edge of a uniform disc and its free end is fixed to the ceiling. The disc moves down unwinding the string with an acceleration equal to (assume string to be vertical)
(a) $\frac{2}{3}$ g (b) $\frac{2}{5}$ g (c) $\frac{2}{7}$ g (d) $\frac{g}{2}$

5. A hoop of radius 3 m weighs 160 kg. It rolls on a horizontal surface so that its centre of mass has a speed 25 cm/s. How much work should be done to stop it?
(a) 10 J (b) 5 J (c) 2.5 J (d) 3.375 J

6. A body is rolling down an inclined plane. If the kinetic energy due to rotation is 40% of kinetic energy due to translation, the body is
(a) a ring (b) a cylinder (c) a hollow sphere (d) a solid sphere

7. A thick hollow sphere rolls down a rough inclined plane without slipping and reaches the bottom with speed v_0, when it is again released on a similar but smooth inclined plane, it reaches the bottom with $\frac{5v_0}{4}$, the radius of gyration of sphere about an axis through its center is (R is the radius of outer surface of the sphere)
(a) $\frac{3R}{5}$ (b) $\frac{3R}{5}$ (c) $\frac{3R}{4}$ (d) $\frac{R}{2}$

8. If moment of Inertia of a solid sphere about any axis passing through its center is I. Then find the moment of inertia of solid sphere about any tangent.
(a) $\frac{7}{2}I$ (b) $\frac{2}{5}I$ (c) $\frac{2}{7}I$ (d) $\frac{5}{2}I$

9. The angular velocity of a body is $\vec{\omega} = \left(2\hat{i} + \hat{j} + 4\hat{k}\right)$rad/s. A torque $\vec{\tau} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right)$N-m acts on it. The rotational power is
(a) 14 W (b) 10 W (c) 15 W (d) 18 W

10. A circular disc of radius R rolls without slipping along the horizontal surface with constant velocity v_0. We consider a point A on the surface of the disc. Then the acceleration of the point A is
(a) constant (b) constant in magnitude
(c) constant in direction (d) constant in magnitude as well as direction

11. A uniform rod AB of mass m and length l is at rest on a smooth horizontal surface. An impulse p is applied to the end B. The time taken by the rod to turn through a right angle is

(a) $2\pi\dfrac{ml}{p}$ (b) $2\pi\dfrac{p}{ml}$ (c) $\dfrac{\pi ml}{12p}$ (d) $\dfrac{\pi p}{ml}$

12. If $\vec{F}$ be a force acting on a particle having the position vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force about the origin, then for equilibrium

(a) $\vec{r}\times\vec{\tau}=0$ and $\vec{F}\times\vec{\tau}=0$ (b) $\vec{r}\times\vec{\tau}=0$ and $\vec{F}\times\vec{\tau}\neq 0$

(c) $\vec{r}\times\vec{\tau}\neq 0$ and $\vec{F}\times\vec{\tau}\neq 0$ (d) $\vec{r}\times\vec{\tau}\neq 0$ and $\vec{F}\times\vec{\tau}=0$

13. Two blocks of masses 10 kg and 4 kg connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of 14 m/s to the heavier block in the direction of the lighter block. The velocity of the centre of mass is

(a) 30 m/s (b) 20 m/s (c) 10 m/s (d) 5 m/s

14. A nucleus moving with a velocity $\vec{v}$ emits an α particle. Let the velocities of the α- particle and the remaining nucleus be $\vec{v}_1$ and $\vec{v}_2$ and their masses be m_1 and m_2. Then

(a) $\vec{v}$, $\vec{v}_1$ and $\vec{v}_2$ must be parallel to each other,

(b) None of the two of $\vec{v}$, $\vec{v}_1$ and $\vec{v}_2$ should be parallel to each other.

(c) $\vec{v}_1+\vec{v}_2$ must be parallel $\vec{v}$

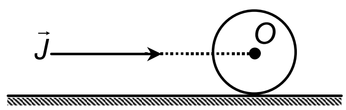
(d) $m_1\vec{v}_1+m_2\vec{v}_2$ must be parallel to $\vec{v}$

15. A uniform sphere is placed on a smooth horizontal surface and a horizontal force F is applied on it at a distance h above the surface. The acceleration of the centre

(a) is maximum when $h = 0$ (b) is maximum when $h = R$

(c) is maximum when $h = 2R$ (d) is independent of h

16. An impulse J is applied on a ring of mass m along a line passing through its centre O and parallel to horizontal surface. The ring is placed on a rough horizontal surface. The linear velocity of centre of ring when it starts rolling without slipping is



(a) J/m (b) $J/2m$ (c) $J/4m$ (d) $J/3m$

17. In a carbon monoxide molecule, the carbon and the oxygen atoms are separated by a distance 1.12×10^{-10} m. The distance of the centre of mass from the carbon atom is

(a) 0.48×10^{-10} m (b) 0.51×10^{-10} m (c) 0.56×10^{-10} m (d) 0.64×10^{-10} m

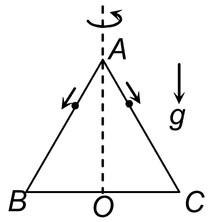
18. A solid spherical ball rolls on an inclined plane without slipping. The ratio of rotational energy and total energy is

(a) $\dfrac{2}{5}$ (b) $\dfrac{2}{7}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{3}{7}$

19. Under a constant torque, the angular momentum of a body changes from A to $4A$ in 4 seconds, the torque is

(a) $\left(\dfrac{3}{4}\right)A$ (b) $\left(\dfrac{1}{4}\right)A$ (c) $3A$ (d) $\left(\dfrac{4}{3}\right)A$

20. An equilateral triangle ABC formed from a uniform wire has two small identical beads initially located at A. The triangle is set rotating about the vertical axis AO. Then the beads are released from rest simultaneously and allowed to slide down, one along AB and the other along AC as shown. Neglecting frictional effects, the quantities that are conserved as the beads slide down, are



(a) angular velocity and total energy (kinetic and potential)
(b) total angular momentum and total energy
(c) angular velocity and moment of inertia about the axis of rotation
(d) total angular momentum and moment of inertia about the axis of rotation

21. Angular momentum of a wheel changes from $2L$ to $5L$ in 3 seconds. Then magnitude of torque acting on it is

(a) $L/5$ (b) $L/3$ (c) $L/2$ (d) L

22. If rotational kinetic energy of a body is ε and its moment of inertia is I, then angular moment can be expressed as

(a) $2\varepsilon I$ (b) $\sqrt{2\varepsilon I}$ (c) $\varepsilon^2/2I$ (d) $2\varepsilon^2 I$

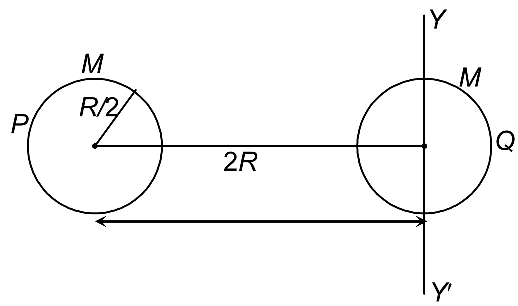
23. Two rings of the same radius and mass are placed such that their centers are at a common point and their planes are perpendicular to each other. The moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings is (mass of the ring = m and radius = r)

(a) $\frac{1}{2}mr^2$ (b) mr^2 (c) $\frac{3}{2}mr^2$ (d) $3mr^2$

24. Three point masses each of mass m are placed at the corners of an equilateral triangle of side b. The moment of inertia of the system about an axis coinciding with one side of triangle is

(a) $3mb^2$ (b) mb^2 (c) $\frac{3}{4}mb^2$ (d) $\frac{2}{3}mb^2$

25. Two spheres each of mass M and radius $R/2$ are connected with a massless rod of length $2R$ as shown in the figure. What will be the moment of inertia of the system about an axis passing through the centre of one of the spheres and perpendicular to the rod?



(a) $\frac{21}{5}MR^2$ (b) $\frac{2}{5}MR^2$

(c) $\frac{5}{2}MR^2$ (d) $\frac{5}{21}MR^2$

EXERCISE – II

IIT-JEE- SINGLE CHOICE CORRECT

1. Consider a system of two identical particles. One of the particles is at rest and the other has an acceleration $\vec{a}$. The centre of mass has an acceleration

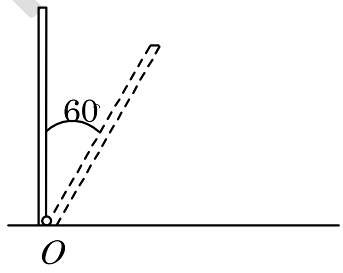
(a) zero (b) $\frac{1}{2}\vec{a}$ (c) $\vec{a}$ (d) $2\vec{a}$

2. A body at rest breaks in two pieces of equal masses. The parts will move
(a) in same direction
(b) along different lines
(c) in opposite directions with equal speeds
(d) in opposite directions with unequal speeds.

3. The moment of inertia of a pair of solid spheres each having mass m and radius r kept in contact about a tangent passing through the point of contact is

(a) $\frac{14}{5}mr^2$ (b) $\frac{7}{5}mr^2$ (c) $\frac{5}{3}mr^2$ (d) $\frac{10}{3}mr^2$

4. A uniform metre stick of mass m is pivoted about a horizontal axis through its lower end O. Initially it is held vertical and is allowed to fall freely down. Its angular velocity at the instant when it makes angle 60° with vertical is

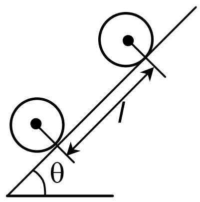


(a) $\sqrt{\frac{g}{2l}}$ (b) $\sqrt{\frac{g}{l}}$

(c) $\sqrt{\frac{3g}{2l}}$ (d) $\sqrt{\frac{2g}{l}}$

5. There are two spheres of the same size, shape and mass. But one is hollow while the other is solid. They are allowed to roll down on an inclined plane –
(a) Solid sphere reaches the bottom of the plane first
(b) Hollow sphere reaches the bottom of the plane first
(c) Both reach the bottom at the same time
(d) Data is insufficient to decide

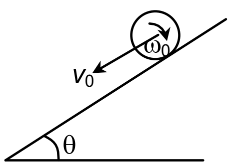
6. A solid sphere of mass m is placed on rough inclined plane as shown in figure. The coefficient of friction μ is insufficient to start pure rolling. The sphere slides a length l on incline from rest and its kinetic energy becomes K. Then work done by friction will be



(a) $\mu mg\cos\theta$ (b) $\mu mg\sin\theta$

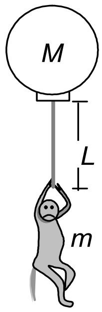
(c) $\frac{2}{5}\mu mg\sin\theta - K$ (d) $K - mgl\sin\theta$

7. A cylinder of mass M and radius R, spinning about its axis with angular velocity ω_0 and having velocity of centre of mass v_0, is placed on a smooth inclined plane as shown in the figure. Then



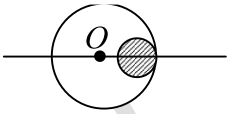
(a) v_0 and ω_0 both always increase.
(b) v_0 always decreases and ω_0 keeps on changing.
(c) v_0 always increases and ω_0 keeps on changing.
(d) v_0 always increases and ω_0 remains constant.

8. The balloon, the light rope and the monkey shown in figure are at rest in the air. If the monkey reaches the top of the rope, by what distance does the balloon descend? Mass of the balloon = M, mass of the monkey = m and the length of the rope ascended by the monkey = L



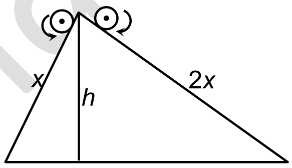
(a) $\frac{mL}{m+M}$ (b) $\frac{ML}{m+M}$
(c) $\frac{mL}{2m+M}$ (d) none

9. From a uniform circular plate of radius R, a small circular plate of radius $R/4$ is cut off as shown. If O is the center of the complete plate, then the distance of the new center of mass of the remaining plate from O will be



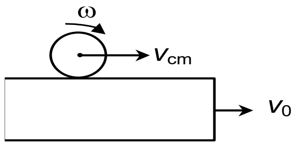
(a) $R/20$ (b) $R/16$ (c) $R/15$ (d) $\frac{3}{4}R$

10. Two identical discs rolls down from top of two identical planes of slant length x and $2x$ but height h is same as shown in figure. The velocities v_1 and v_2 acquired by the disc, when they reach the bottom of the incline, are related as



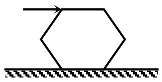
(a) $v_1 = v_2$ (b) $v_1 = 2v_2$
(c) $2v_1 = v_2$ (d) none of these

11. in the given figure, the sphere rolls without slipping on the plank which is moving with constant velocity v0. the radius and angular velocity of the sphere is r and ω respectively. the velocity of centre of mass of the sphere is



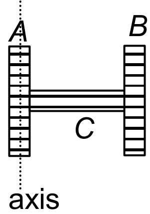
(a) $v_0 + r\omega$ (b) $v_0 - r\omega$
(c) rω (d) v0

12. For the toppling of the shown regular hexagon, the coefficient of friction must be



(a) > 0.29 (b) < 0.29
(c) = 0.29 (d) ≤ 0.29

13. Three identical rods each of length L and mass M joined together to form a letter H. What is the moment of inertia of the system about one of the sides of H as shown in figure?

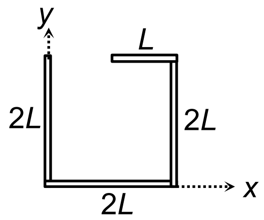


(a) $\frac{ML^2}{3}$ (b) $\frac{ML^2}{4}$
(c) $\frac{2ML^2}{3}$ (d) $\frac{4ML^2}{3}$

14. An inclined plane makes an angle of 30^0 with the horizontal. A solid sphere rolling down this inclined plane from rest without slipping has a linear acceleration equal to

(a) $g/3$ (b) $2g/3$ (c) $5g/7$ (d) $5g/14$

16. A uniform rod of length $7L$ is bent in the shape as shown in the figure. The co-ordinates of the centre of mass of the system are



(a) $\frac{15}{7}L, \frac{6}{7}L$ (b) $\frac{15}{14}L, \frac{6}{7}L$

(c) $\frac{15}{7}L, \frac{6}{14}L$ (d) $\frac{15}{14}L, \frac{6}{14}L$

17. A cannon shell is fired to hit a target at a horizontal distance R. However it breaks into two equal parts at its highest point. One part returns to the cannon. The other part

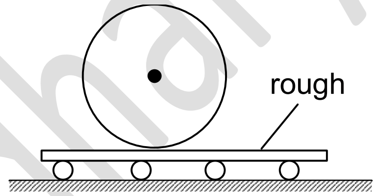
(a) will fall at a distance R beyond target

(b) will fall at a distance $3R$ beyond target

(c) will hit the target

(d) will fall at a distance $2R$ beyond target

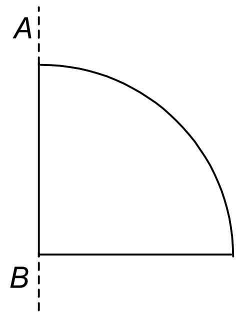
18. The plank in the figure moves a distance 100 mm to the right while the center of mass of the sphere of radius 150 mm moves a distance 75 mm to the left. The angular displacement of the sphere (in radian) is (there is no slipping anywhere)



(a) $\frac{1}{6}$ (b) $\frac{7}{6}$

(c) 1 (d) $\frac{1}{2}$

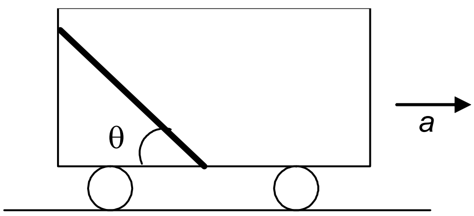
19. A lamina of mass M is in shape of a quarter of circle of radius R as shown in figure. The moment of inertia of this lamina about axis AB is



(a) $\frac{MR^2}{8}$ (b) $\frac{MR^2}{4}$

(c) $\frac{MR^2}{2}$ (d) MR^2

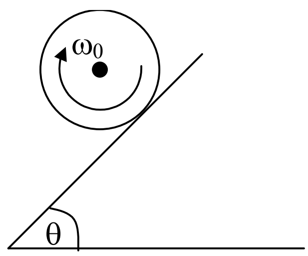
20. A smooth rod of length □ is kept inside a trolley at an angle θ as shown in the figure. What should be the acceleration a of the trolley so that the rod remains in equilibrium with respect to it ?



(a) $g \tan\theta$ (b) $g \cos\theta$

(c) $g \sin\theta$ (d) $g \cot\theta$

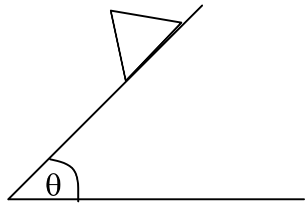
21. A uniform cylinder of radius R is spinned to an angular velocity ω_0 and then placed on an incline for which coefficient of friction is $\mu = \tan\theta$. (θ is the angle of incline). The center of mass of the cylinder will remain stationary for time



(a) $\frac{R\omega_0}{g\sin\theta}$ (b) $\frac{R\omega_0}{3g\sin\theta}$

(c) $\frac{R\omega_0}{2g\sin\theta}$ (d) $\frac{3R\omega_0}{2g\sin\theta}$

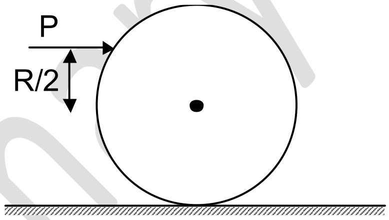
22. A block having equilateral triangular cross-section of side a and mass m is placed on a rough inclined surface, so that it remains in equilibrium as shown in figure. The torque of normal force acting on the block about its center of mass is



(a) $\frac{\sqrt{3}}{2}mga\sin\theta$ (b) $\frac{1}{2\sqrt{3}}mga\sin\theta$

(c) $\frac{1}{2\sqrt{3}}mga\cos\theta$ (d) zero

23. A cylinder of mass M and radius R lying on a rough horizontal surface for which coefficient of friction is μ is pushed by applying a horizontal force P at a distance $R/2$ from the center as shown in figure. The frictional force acting at the contact is



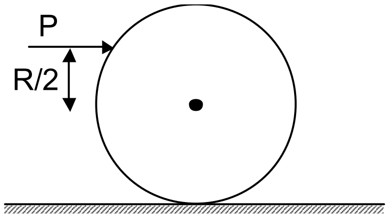
(a) μMg towards right

(b) μMg towards left

(c) less than μ Mg towards right

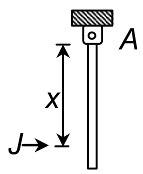
(d) zero

23. A cylinder of mass M and radius R lying on a rough horizontal surface for which coefficient of friction is μ is pushed by applying a horizontal force P at a distance $R/2$ from the center as shown in figure. The frictional force acting at the contact is



(a) μMg towards right
(b) μMg towards left
(c) less than μ Mg towards right
(d) zero

24. A uniform rod of length l is pivoted at point A. It is struck by a horizontal force which delivers an impulse J at a distance x from point A as shown in figure, if impulse delivered by pivot is zero at initially then x is equal to



(a) $\frac{l}{2}$ (b) $\frac{l}{3}$ (c) $\frac{2l}{3}$ (d) $\frac{3l}{4}$

25. A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the center of the face, at a height $\frac{3a}{4}$ above the base. What is the minimum value of F for which the cube begins to tip about an edge?

(a) $\frac{2}{3}mg$ (b) mg (c) $\frac{3}{2}mg$ (d) $2\,mg$

EXERCISE – III

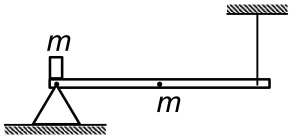
ONE OR MORE THAN ONE CHOICE CORRECT

1. If a system is in rotational equilibrium, the net torque acting on it must balance. This is true only if the torques are taken about
(a) the centre of the mass of the system if centre of mass is accelerated
(b) any point on the system
(c) the centre of the mass of the system if centre of mass is non-accelerated
(d) any point on the system or outside it if centre of mass is non-accelerated.

2. Two identical spheres are placed on a rough horizontal surface. The sphere A is in pure rolling with linear velocity v and B is at rest. If friction is absent between the spheres, after elastic collision
(a) A will be linearly at rest and B will move linearly with velocity v just after collision.
(b) both will have pure rolling motion with linear velocity in same direction after some time.
(c) both will move in opposite direction with same speed
(d) both will move in opposite direction with different speed

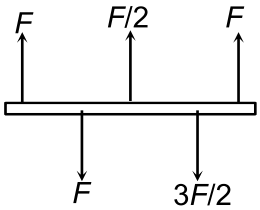
3. A sphere can roll on
(a) a smooth horizontal surface
(b) a smooth inclined surface
(c) a rough horizontal surface
(d) a rough inclined surface

4. A uniform rod of mass m is attached by a massless string at one end and at other end it is supported with a point of mass m placed at this end as shown, then



(a) tension in the string is $\frac{mg}{2}$
(b) tension in the string is $\frac{2mg}{3}$
(c) reaction force from the support is $\frac{3mg}{2}$
(d) reaction force from the support is $\frac{4mg}{3}$

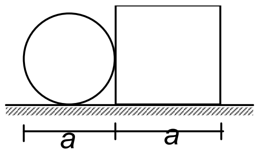
5. Five forces are acting on a uniform rod placed on smooth horizontal surface at a equal distances as shown, then



(a) rod will move up or down depending on value of F.
(b) rod will be in translatory equilibrium.
(c) rod will rotate anticlockwise.
(d) rod will rotate clockwise.

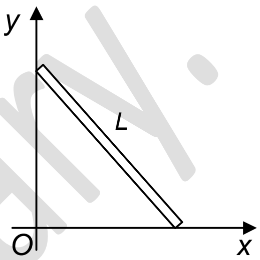
6. A uniform heavy disc is rotating at constant angular velocity ω about a vertical axis through its centre and perpendicular to the plane of the disc. Let L be its angular momentum. A particle is dropped vertically on the disc and sticks to it. Then
(a) ω will change.
(b) L will change.
(c) ω will not change
(d) L will not change

7. A circular plate of diameter a is kept in contact with a square plate of edge a as shown in figure. The density of the material and the thickness are same everywhere. Then



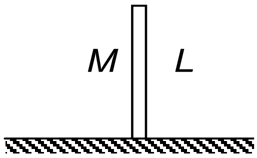
(a) x-coordinate of the centre of mass will lie inside the square plate.
(b) x-coordinate of the centre of mass will lie inside the circular plate.
(c) y-coordinate of the centre of mass will lie inside the circular plate.
(d) y-coordinate of the centre of mass will lie inside the square plate.

8. A rod of length L leans by its upper end against a smooth vertical wall while its other end is on a smooth floor. The end that leans against the wall moves uniformly downward. Select the incorrect alternative



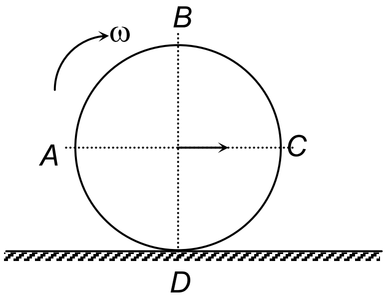
(a) The speed of lower end increases at a constant rate
(b) The speed of the lower end gets smaller and smaller and vanishes when the upper end touches the ground.
(c) The speed of the lower end decreases but never becomes zero
(d) The speed of lower end first increases then decreases.

9. A uniform rod of mass M and length L is held vertically on a smooth horizontal surface. When the rod is released, choose the correct alternative

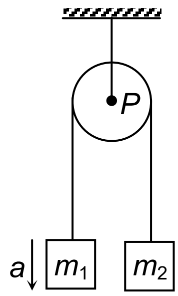


(a) The center of mass of the rod accelerated in the vertical direction
(b) Initially, the magnitude of the normal reaction is Mg
(c) When the rod becomes just horizontal, the magnitude of the normal reaction becomes $Mg/2$
(d) When the rod becomes just horizontal, the magnitude of the normal reaction becomes $Mg/4$

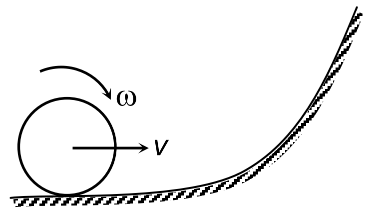
10. A uniform ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure,
(a) section ABC has greater kinetic energy than section ADC
(b) section BC has greater kinetic energy than section CD
(c) section BC has the same kinetic energy as DA
(d) the section AB, BC, CD and DA have same kinetic energy.



11. In the figure, the blocks have unequal masses m_1 and m_2 $(m_1 > m_2)$. Block of mass m_1 has downward acceleration a. The pulley P has a radius r and some mass. The string does not slip on the pulley.
(a) The two sections of the string have same tension
(b) The two blocks have acceleration of equal magnitude
(c) $a < \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) g$
(d) Angular acceleration of the pulley is $\dfrac{2a}{r}$

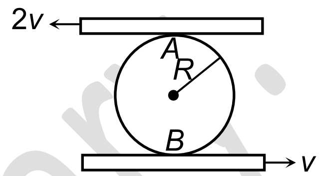


12. A solid sphere moving without slipping on a horizontal rough surface and start rising on inclined rough surface then



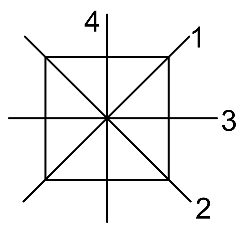
(a) friction force is zero when moving on horizontal surface
(b) direction of friction force is upward when moving upward on inclined plane
(c) direction of friction force is upward when moving downward on inclined plane.
(d) friction force is always opposite to the motion of the sphere.

13. A disc of the radius R is confined to roll without slipping at A and B. If the plates have the velocities v and $2v$ as shown. Then



(a) the linear velocity of the disc is $\frac{v}{2}$ towards left.
(b) the linear velocity of the disc is $\frac{3v}{2}$ towards left.
(c) the angular velocity of the disc is $\frac{3v}{2R}$ anticlockwise.
(d) the angular velocity of the disc is $\frac{3v}{2R}$ clockwise.

14. The moment of inertia of a thin square plate $ABCD$ of uniform thickness about an axis passing through the centre O and perpendicular to plane of plate is (I_1, I_2, I_3, I_4 are the moment of inertias of the plate about the shown axis)



(a) $I_1 + I_2$ (b) $I_3 + I_4$
(c) $I_1 + I_4$ (d) $I_2 + I_3$

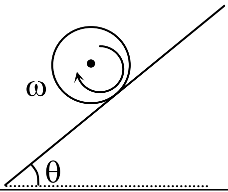
15. The torque $\vec{\tau}$ on a body about a given point is found to be equal to $\vec{A} \times \vec{L}$, where $\vec{A}$ is a constant vector and $\vec{L}$ is the angular momentum of the body about that point. Then
(a) $\frac{d\vec{L}}{dt}$ is perpendicular to $\vec{L}$ at all instants of time.
(b) the component of $\vec{L}$ in the direction of $\vec{A}$ does not change with time.
(c) the magnitude of $\vec{L}$ does not change with time.
(d) $\vec{L}$ does not change with time.

EXERCISE – IV

MATCH THE FOLLOWING

Note: Each statement in column – I has one or more than one match in column –II.

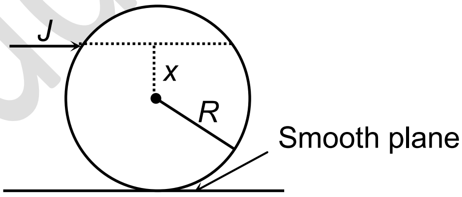
1. The diagram shows a rough inclined plane with inclination θ. A horizontal disc rotating with angular velocity ω (clockwise) is gently placed on the rough plane.



Column-I	Column-II
I. If the coefficient of friction is tanθ	**A.** Linear acceleration of the disc is zero.
II. If coefficient of friction is less than tanθ	**B.** Linear acceleration of the disc is downward along the inclined plane.
III. If coefficient of friction is greater than tanθ	**C.** Linear acceleration of the disc is upward along the inclined plane.
IV. If coefficient of friction is zero	**D.** Friction force is upward along the inclined plane.

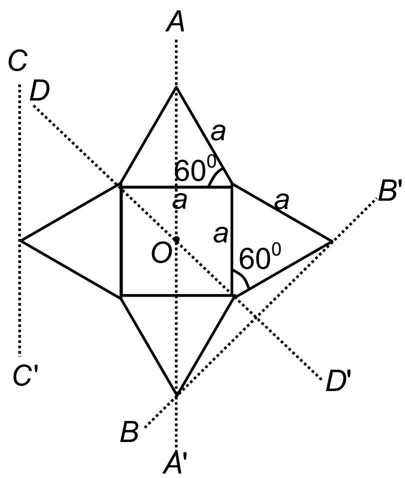
Note: Each statement in column – I has only one match in column –II

2. If a body like ring, disc or sphere kept on a smooth plane is given a impulse at some particular point, it will start pure rolling. If vertical distance of that particular point from centre of mass is x_1 for ring, x_2 for disc, x_3 for solid sphere and x_4 for hollow sphere, each of radius R, then match the x_1, x_2, x_3 and x_4 to their corresponding values in given columns.



Column-I	Column-II
I. x_1	**A.** $2/5\ R$
II. x_2	**B.** R
III. x_3	**C.** $2/3\ R$
IV. x_4	**D.** $R/2$

3. A symmetric plate of mass M and shape as shown in figure, has moment of inertia I about axis AA'. If moment of inertia of this plate about axis passing through it centre O and perpendicular to the plane of plate is I_1 and moment of inertia about axis BB', CC' and DD' are I_2, I_3 and I_4 respectively then match the following



Column-I	Column-II
I. I_1	**A.** $I + \left(\frac{2+\sqrt{3}}{4}\right)Ma^2$
II. I_2	**B.** I
III. I_3	**C.** $2I$
IV. I_4	**D.** $I + \left(\frac{2+\sqrt{3}}{2}\right)Ma^2$

REASONING TYPE

Directions: Read the following questions and choose

(A) If both the statements are true and statement-2 is the correct explanation of **statement-1**.
(B) If both the statements are true but statement-2 is not the correct explanation of **statement-1**.
(C) If statement-1 is True and **statement-2** is False.
(D) If statement-1 is False **and statement-2** is True.

1. **Statement-1**: Work done by frictional force on a sphere rolling without slipping on an inclined plane is negative.

Statement-2: Work done by the force F, $W = \int \vec{F}.d\vec{S}$

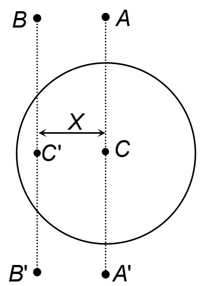
(a) (A) (b) (B) (c) (C) (d) (D)

2. **Statement-1**: Five charged particles having equal charge are released in gravity free space the centre of mass of the arrangement will move with some acceleration.

Statement-2: Acceleration of centre of mass of the system $= \dfrac{\vec{F}_{\text{net on the system}}}{\text{total mass of the system}}$

(a) (A) (b) (B) (c) (C) (d) (D)

3. **Statement-1**: If the moment of inertia of a non-uniform ring of mass M and radius R is known about an AA', then we can calculate the moment of inertia about BB' (C is the geometrical centre of ring)

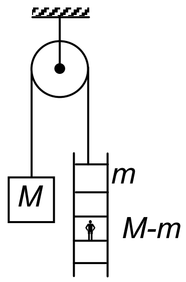


Statement-2: $I_{BB'} = I_{cm} + MX^2$

(a) (A) (b) (B)
(c) (C) (d) (D)

4. **Statement-1**: A massless rope thrown over a frictionless pulley has a ladder with a mass of m on one its ends and a counter balancing mass M on its other end. If a man climbs with some velocity then centre of mass of the system (man, ladder and balancing mass) moving upward.

Statement-2: Net force on the system is in upward direction.



(a) (A) (b) (B) (c) (C) (d) (D)

5. **Statement-1**: Two balls are thrown simultaneously in air. The acceleration of centre of mass of the two balls while in air depends on the masses of the two balls.

Statement-2: The acceleration of centre of mass is given by $\vec{a} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$

(a) (A) (b) (B) (c) (C) (d) (D)

LINKED COMPREHENSION TYPE

A thin uniform square plate of side l and mass m can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of same mass m flying with velocity v at right angles to the plate strikes at a distance x from the vertical axis and plate takes a time $\dfrac{7\pi l}{6v}$ in one complete revolution after collision. Speed of the particle after collision is $\dfrac{v}{7}$ in opposite direction as before.

1. Find the value of x.

(a) $\dfrac{l}{3}$ (b) $\dfrac{7l}{12}$

(c) $\dfrac{l}{2}$ (d) $\dfrac{3l}{4}$

2. Total kinetic energy lost during collision is

(a) $\dfrac{1}{4}mv^2$ (b) $\dfrac{1}{8}mv^2$

(c) $\dfrac{7}{16}mv^2$ (d) zero

3. The horizontal component of the resultant force which the axis will exert on the plate after the collision is

(a) $\dfrac{72mv^2}{49l}$ (b) $\dfrac{8mv^2}{7l}$

(c) $\dfrac{9mv^2}{7l}$ (d) zero

4. The following are conserved in this collision:

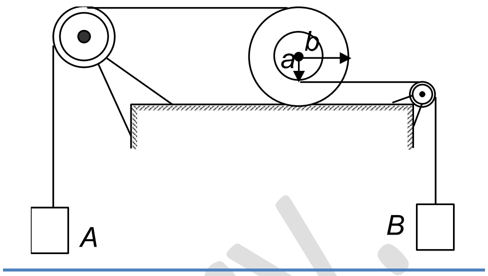
(a) linear momentum (b) kinetic energy

(c) angular momentum (d) kinetic energy and angular momentum

EXERCISE – V

SUBJECTIVE PROBLEMS

1. In the arrangement shown in figure mass of blocks A and B is $m_1 = 0.5$ kg and $m_2 = 10$ kg, respectively and mass of spool is $M = 8$ kg. Inner and outer radii of the spool are $a = 10$ cm and $b = 15$ cm respectively. Its moment of inertia about its own axis is $I_0 = 0.10$ kgm^2. If friction be sufficient to prevent sliding, calculate acceleration of blocks A and B. ($g = 10$ m/s^2)

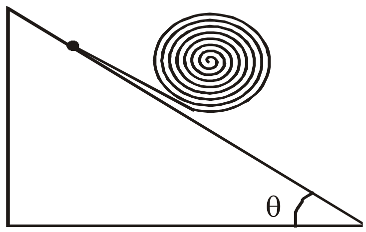


2. An inelastic uniform sphere of radius a is sliding without rotation on a smooth horizontal plane when it impinges on a thin horizontal rod at right angles to its direction of motion and at a height b from the plane. Show that it will just roll over the rod if its velocity be $\left(\frac{a}{a-b}\right)\sqrt{\frac{14gb}{5}}$.

3. A thin uniform square plate with side ℓ and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity v at right angles to the plates strikes elastically its centre. Find (a) the velocity of ball v' after impact; (b) the horizontal component of the resultant force which the axis will exert on the plate after impact.

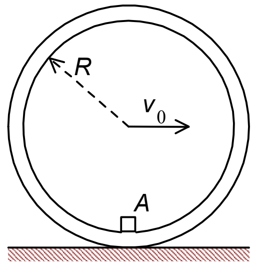
4. A smooth uniform rod AB of mass M and length ℓ rotates freely with angular velocity ω_0 in a horizontal plane about a stationary vertical axis passing through its end A. A small sleeve of mass m starts sliding along the rod from A. Find the velocity v' of the sleeve relative to rod at the moment it reaches the other end B.

5. A length L of flexible tape is tightly wound. It is then allowed to unwind as it rolls down an incline that makes an angle θ with the horizontal, the upper end of the tape being fixed. Find the time taken by the tape to unwind completely.



6. A thin rod is held resting on the ground with its length inclined at an angle α to the horizontal. The coefficient of friction between rod and ground is μ. Show that when the rod is let go it will start slipping on the ground if $\mu < \frac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}$.

7. A small body A is fixed to the inside of a thin rigid hoop of radius R and mass equal to that of the body A. The hoop rolls without slipping over a horizontal plane. At the moment when the body A gets into the lower position the centre of the hoop moves with velocity v_0. At what values of v_0 will the hoop move without bouncing?

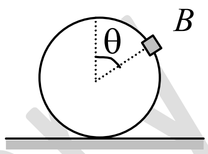


8. A small body of mass m is attached at B to a hoop of mass $3m$ and radius r. The system is released from rest with $\theta = 90°$ and rolls without sliding. Determine

(a) the angular acceleration of the hoop.

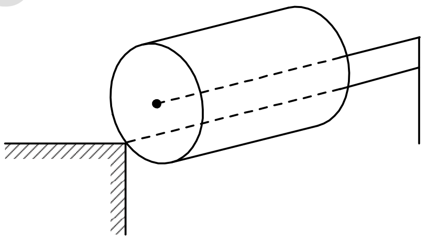
(b) the horizontal and vertical components of the acceleration of B.

(c) normal reaction and frictional force just after the release.



9. A solid sphere rolling on a rough horizontal surface with linear speed v collides elastically with a fixed, smooth, vertical wall. Find the speed of the sphere after it has started pure rolling in the backward direction.

10. A rectangular rigid fixed block has a long horizontal edge. A solid homogeneous cylinder of radius R is placed horizontally at rest with its length parallel to the edge such that the axis of the cylinder and the edge of the block are in the same vertical plane as shown in Figure. There is sufficient friction present at the edge so that a very small displacement causes the cylinder to roll off the edge without slipping. Determine.



(a) the angle θ_C through which the cylinder rotates before it leaves contact with edge

(b) the speed of the centre of mass of cylinder before leaving contact with edge and the ratio of translational to rotational kinetic energies of the cylinder when its centre of mass is in horizontal line with the edge.

ANSWERS

EXERCISE – I

NEET-SINGLE CHOICE CORRECT

1. (c)	2. (b)	3. (d)	4. (a)	5. (a)
6. (d)	7. (c)	8. (a)	9. (d)	10. (b)
11. (c)	12. (a)	13. (c)	14. (d)	15. (d)
16. (b)	17. (d)	18. (b)	19. (a)	20. (b)
21. (d)	22. (b)	23. (c)	24. (c)	25. (a)

EXERCISE – II

IIT-JEE-SINGLE CHOICE CORRECT

1. (b)	2. (c)	3. (a)	4. (c)	5. (a)
6. (d)	7. (d)	8. (a)	9. (a)	10. (a)
11. (a)	12. (a)	13. (d)	14. (d)	15. (b)
16. (b)	17. (a)	18. (b)	19. (b)	20. (d)
21. (c)	22. (b)	23. (d)	24. (c)	25. (a)

EXERCISE – III

ONE OR MORE THAN ONE CHOICE CORRECT

1. (a,d)	2. (a,b)	3. (a,c,d)	4. (a,c)	5. (b,d)
6. (a,d)	7. (a,d)	8. (a,c,d)	9. (a,b,d)	10. (a,b)
11. (b,c)	12. (a,b,c)	13. (a,c)	14. (a,b,c,d)	15. (a,b,c)

EXERCISE – IV

MATCH THE FOLLOWING

1. I – A, D; II – B, D; III – C, D; IV – B

2. I – B ; II – D ; III – A ; IV – C

3. I – C ; II – A ; III – D ; IV – B

REASONING TYPE

1. (d)	2. (d)	3. (d)	4. (a)	5. (d)

LINKED COMPREHENSION TYPE

1. (c)	2. (d)	3. (a)	4. (b)	

EXERCISE – V

SUBJECTIVE PROBLEMS

1. $\boldsymbol{a_A = 3\text{ms}^{-2}}$ **(upward)**

$a_B = 0.5\text{ ms}^{-2}$ (downward)

3. (a) $v' = \dfrac{3m-4M}{3m+4M}v$; (b) $F = \dfrac{8Mv^2}{\ell\left[1+\dfrac{4M}{3m}\right]^2}$

4. $v' = \dfrac{\omega_0 \ell}{\sqrt{1+\dfrac{3m}{M}}}$

5. $\sqrt{\dfrac{3L}{g\sin\theta}}$

7. $v_{0(\text{max})} = \sqrt{8gR}$

8. (a) $\dfrac{g}{8r}$; (b) $\dfrac{g}{8}$ and $\dfrac{g}{8}$; (c) 3.88 mg, 0.5 mg

9. $\dfrac{3}{7}v$

10. (a) $\theta_C = \cos^{-1}\left(\dfrac{4}{7}\right)$; (b) $\sqrt{\dfrac{4}{7}gR}$, 6